

EC5555
Economics Masters Refresher Course in Mathematics
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Lecture 7 – Optimization with inequality constraints
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Optimization with inequality constraints: the Kuhn-Tucker (KT) conditions

The **KT conditions** for the problem

$$\max_x f(x) \text{ subject to } g_j(x) \leq c_j \text{ for } j = 1, \dots, m$$

are

$$L'_i(x) = 0 \text{ for } i = 1, \dots, n$$

$$\lambda_j \geq 0, g_j(x) \leq c_j \text{ and } \lambda_j[g_j(x) - c_j] = 0 \text{ for } j = 1, \dots, m,$$

where

$$L(x) = f(x) - \sum_{j=1}^m \lambda_j (g_j(x) - c_j).$$

Necessity and sufficiency of KT conditions

- A) The KT conditions are both necessary and sufficient if the objective function is concave and
- **either** each constraint is linear
 - **or** each constraint function is convex and some vector of the variables satisfies all constraints strictly.
- B) Suppose that the objective function is twice differentiable and quasiconcave and every constraint is linear.
- If x^* solves the problem then there exists a unique vector λ such that (x^*, λ) satisfies the Kuhn-Tucker conditions, and
 - if (x^*, λ) satisfies the Kuhn-Tucker conditions and $f'_i(x^*) \neq 0$ for $i = 1, \dots, n$ then x^* solves the problem.

Example

$$\begin{aligned} \max_{x_1, x_2} & \quad [-(x_1 - 4)^2 - (x_2 - 4)^2] \\ \text{s.t.} & \quad x_1 + x_2 \leq 4 \text{ and } x_1 + 3x_2 \leq 9 \end{aligned}$$

The objective function is concave and the constraints are both linear, so the solutions of the problem are the solutions of the Kuhn-Tucker conditions.

Kuhn Tucker conditions are

$$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$

$$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$

$$x_1 + x_2 \leq 4, \lambda_1 \geq 0, \text{ and } \lambda_1(x_1 + x_2 - 4) = 0$$

$$x_1 + 3x_2 \leq 9, \lambda_2 \geq 0, \text{ and } \lambda_2(x_1 + 3x_2 - 9) = 0$$

To solve this system of condition we have to consider all possibilities about the values of lambdas

We have to consider the following 4 cases:

1) $\lambda_1 = \lambda_2 = 0$

2) $\lambda_1 > 0 \lambda_2 = 0$

3) $\lambda_1 = 0 \lambda_2 > 0$

4) $\lambda_1 > 0 \lambda_2 > 0$

Kuhn Tucker conditions are

$$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$

$$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$

$$x_1 + x_2 \leq 4, \lambda_1 \geq 0, \text{ and } \lambda_1(x_1 + x_2 - 4) = 0$$

$$x_1 + 3x_2 \leq 9, \lambda_2 \geq 0, \text{ and } \lambda_2(x_1 + 3x_2 - 9) = 0$$

Case 1: $\lambda_1 = \lambda_2 = 0$

KT conditions are

$$-2(x_1 - 4) = 0$$

$$-2(x_2 - 4) = 0$$

$$x_1 + x_2 \leq 4,$$

$$x_1 + 3x_2 \leq 9$$

Then $x_1 = 4$ and $x_2 = 4$

It not a solution because the last two inequalities are not satisfied

Kuhn Tucker conditions are

$$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$

$$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$

$$x_1 + x_2 \leq 4, \lambda_1 \geq 0, \text{ and } \lambda_1(x_1 + x_2 - 4) = 0$$

$$x_1 + 3x_2 \leq 9, \lambda_2 \geq 0, \text{ and } \lambda_2(x_1 + 3x_2 - 9) = 0$$

Case 2: $\lambda_1 > 0$ $\lambda_2 = 0$

KT conditions are

$$-2(x_1 - 4) - \lambda_1 = 0$$

$$-2(x_2 - 4) - \lambda_1 = 0$$

$$x_1 + x_2 - 4 = 0$$

$$x_1 + 3x_2 \leq 9,$$

From the first 2 equations $x_1 = x_2$

Using the third equation we get $x_1 = x_2 = 2$

It is a solution because the last inequality is satisfied

Kuhn Tucker conditions are

$$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$

$$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$

$$x_1 + x_2 \leq 4, \lambda_1 \geq 0, \text{ and } \lambda_1(x_1 + x_2 - 4) = 0$$

$$x_1 + 3x_2 \leq 9, \lambda_2 \geq 0, \text{ and } \lambda_2(x_1 + 3x_2 - 9) = 0$$

Case 3: $\lambda_1 = 0$ $\lambda_2 > 0$

KT conditions are

$$-2(x_1 - 4) - \lambda_2 = 0$$

$$-2(x_2 - 4) - 3\lambda_2 = 0$$

$$x_1 + x_2 \leq 4$$

$$x_1 + 3x_2 - 9 = 0$$

From the first 2 equations $x_2 = 3x_1 - 8$

Using the last equation we get $x_1 = 3.3$

It not a solution because it not satisfy the inequality

Kuhn Tucker conditions are

$$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$

$$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$

$$x_1 + x_2 \leq 4, \lambda_1 \geq 0, \text{ and } \lambda_1(x_1 + x_2 - 4) = 0$$

$$x_1 + 3x_2 \leq 9, \lambda_2 \geq 0, \text{ and } \lambda_2(x_1 + 3x_2 - 9) = 0$$

Case 4: $\lambda_1 > 0$ $\lambda_2 > 0$

KT conditions are

$$-2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$

$$-2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$

$$x_1 + x_2 - 4 = 0$$

$$x_1 + 3x_2 = 0$$

Using the last two equations we get $x_1 = 1.5$ and $x_2 = 2.5$

Replacing in the first two equations we get the values of lambdas

$$\lambda_1 = 6 \quad \lambda_2 = -1$$

This is not a solution because it violates the condition $\lambda_2 \geq 0$.

Optimization with inequality constraints: non negativity constraints

The general form of such a problem is:

$$\begin{aligned} & \max_x f(x) \text{ subject to} \\ & g_j(x) \leq c_j \text{ for } j = 1, \dots, m \text{ and} \\ & x_i \geq 0 \text{ for } i = 1, \dots, n. \end{aligned}$$

Lagrangean is

$$L(x) = f(x) - \sum_{j=1}^m \lambda_j (g_j(x) - c_j) - \sum_{j=1}^n \lambda_{m+j} (-x_j)$$

It is a special case of the general maximization problem with inequality constraints: the nonnegativity constraint on each variable is simply an additional inequality constraint.

Specifically, if we define the function g_{m+i} for $i = 1, \dots, n$ by $g_{m+i}(x) = -x_i$ and let $c_{m+i} = 0$ for $i = 1, \dots, n$, then we may write the problem as

$$\begin{aligned} & \max_x f(x) \text{ subject to} \\ & g_j(x) \leq c_j \text{ for } j = 1, \dots, m+n \end{aligned}$$

and solve it using the Kuhn-Tucker conditions

Optimization with inequality constraints: non negativity constraints

Approaching the problem in this way involves working with $n + m$ Lagrange multipliers, which can be difficult if n is large.

Then we can use an alternative approach, the ***modified Lagrangean***

Consider the following problem:

$$\begin{aligned} & \max_x f(x) \text{ subject to} \\ & g_j(x) \leq c_j \text{ for } j = 1, \dots, m \text{ and} \\ & x_i \geq 0 \text{ for } i = 1, \dots, n. \end{aligned}$$

The *modified Lagrangean* is:

$$M(x) = f(x) - \sum_{j=1}^m \lambda_j (g_j(x) - c_j)$$

Kuhn-Tucker conditions for the modified Lagrangean:

$$M'_i(x) \leq 0, \quad x_i \geq 0, \quad \text{and } x_i \cdot M'_i(x) = 0 \text{ for } i = 1, \dots, n$$

$$g_j(x) \leq c_j, \quad \lambda_j \geq 0, \quad \text{and } \lambda_j \cdot [g_j(x) - c_j] = 0 \text{ for } j = 1, \dots, m.$$

in any problem for which the original Kuhn-Tucker conditions may be used, we may alternatively use the conditions for the modified Lagrangean.

For most problems in which the variables are constrained to be nonnegative, the Kuhn-Tucker conditions for the modified Lagrangean are easier than the conditions for the original Lagrangean

Example.

Consider the problem

$$\max_{x,y} xy \text{ subject to } x + y \leq 6, x \geq 0, \text{ and } y \geq 0$$

Function xy is twice-differentiable and quasiconcave and the constraint functions are linear, so the Kuhn-Tucker conditions are necessary and if $((x^*, y^*), \lambda^*)$ satisfies these conditions and no partial derivative of the objective function at (x^*, y^*) is zero then (x^*, y^*) solves the problem.

Solutions of the Kuhn-Tucker conditions at which all derivatives of the objective function are zero may or may not be solutions of the problem

We try to solve it

- 1) using the lagrangean
- 2) Using the modified lagrangean

1) Using Lagrangean

$$L(x, y) = xy - \lambda_1(x + y - 6) - \lambda_2(-x) - \lambda_3(-y)$$

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) = 0$$

We have to consider the following 8 cases:

1) $\lambda_1 = 0 \lambda_2 = 0 \lambda_3 = 0$

2) $\lambda_1 > 0 \lambda_2 = 0 \lambda_3 = 0$

3) $\lambda_1 = 0 \lambda_2 > 0 \lambda_3 = 0$

4) $\lambda_1 > 0 \lambda_2 > 0 \lambda_3 = 0$

5) $\lambda_1 = 0 \lambda_2 = 0 \lambda_3 > 0$

6) $\lambda_1 > 0 \lambda_2 = 0 \lambda_3 > 0$

7) $\lambda_1 = 0 \lambda_2 > 0 \lambda_3 > 0$

8) $\lambda_1 > 0 \lambda_2 > 0 \lambda_3 > 0$

Kuhn Tucker conditions are:

$$\begin{aligned}y - \lambda_1 + \lambda_2 &= 0 \\x - \lambda_1 + \lambda_3 &= 0 \\ \lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) &= 0 \\ \lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) &= 0 \\ \lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) &= 0\end{aligned}$$

Case 1: $\lambda_1 = 0$ $\lambda_2 = 0$ $\lambda_3 = 0$

Kuhn Tucker conditions are:

$$\begin{aligned}y &= 0 \\x &= 0 \\ \lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) &= 0 \\ \lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) &= 0 \\ \lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) &= 0\end{aligned}$$

All conditions are satisfied, but the first derivatives of the objective function, evaluated at $x=y=0$ are equal to zero. Then this could be a solution.

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) = 0$$

Consider now $\lambda_1 = 0$

Kuhn Tucker conditions are:

$$y + \lambda_2 = 0$$

$$x + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) = 0$$

Then $\lambda_2 = -y$ and $x = -\lambda_3$. If λ_2 (λ_3) is strictly positive, then y (x) is strictly negative and does not satisfy the last two conditions.

This allows us to eliminate all combinations where $\lambda_1 = 0$ and at least one among λ_2 and λ_3 is strictly positive, then combinations 3, 5, 7

Then we have to check only the combinations 2, 4, 6, 8

Kuhn Tucker conditions are:

$$\begin{aligned}y - \lambda_1 + \lambda_2 &= 0 \\x - \lambda_1 + \lambda_3 &= 0 \\ \lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) &= 0 \\ \lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) &= 0 \\ \lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) &= 0\end{aligned}$$

Case 2) $\lambda_1 > 0$ $\lambda_2 = 0$ $\lambda_3 = 0$

Kuhn Tucker conditions are:

$$\begin{aligned}y - \lambda_1 &= 0 \\x - \lambda_1 &= 0 \\ \lambda_1 \geq 0, x + y &= 6, \\ (-x) &\leq 0, \\ (-y) &\leq 0,\end{aligned}$$

From the first 3 conditions we have that $x = y = 3$ and $\lambda_1 = 3$

These values satisfy the last conditions and the derivatives of objective function evaluated in this point are different from zero.

Kuhn Tucker conditions are:

$$\begin{aligned}y - \lambda_1 + \lambda_2 &= 0 \\x - \lambda_1 + \lambda_3 &= 0 \\ \lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) &= 0 \\ \lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) &= 0 \\ \lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) &= 0\end{aligned}$$

Case 4) $\lambda_1 > 0$ $\lambda_2 > 0$ $\lambda_3 = 0$

Kuhn Tucker conditions are:

$$\begin{aligned}y - \lambda_1 + \lambda_2 &= 0 \\x - \lambda_1 &= 0 \\ \lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) &= 0 \\ (-x) = 0, \quad \lambda_2(-x) &= 0 \\ (-y) \leq 0\end{aligned}$$

From condition in the 4th line we have $x = 0$,

replacing in the second line we get $\lambda_1 = 0$, a contradiction with the initial assumption of $\lambda_1 > 0$

Kuhn Tucker conditions are:

$$\begin{aligned}y - \lambda_1 + \lambda_2 &= 0 \\x - \lambda_1 + \lambda_3 &= 0 \\ \lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) &= 0 \\ \lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) &= 0 \\ \lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) &= 0\end{aligned}$$

Case 6) $\lambda_1 > 0$ $\lambda_2 = 0$ $\lambda_3 > 0$

The first two conditions are

$$\begin{aligned}y - \lambda_1 &= 0 \\x - \lambda_1 + \lambda_3 &= 0\end{aligned}$$

and the last is implies $y=0$.

Replacing it in the first line we find that $\lambda_1 = 0$, a contradiction with the initial assumption of $\lambda_1 > 0$

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$\lambda_1 \geq 0, \quad x + y \leq 6, \quad \lambda_1(x + y - 6) = 0$$

$$\lambda_2 \geq 0, \quad (-x) \leq 0, \quad \lambda_2(-x) = 0$$

$$\lambda_3 \geq 0, \quad (-y) \leq 0, \quad \lambda_3(-y) = 0$$

Case 8) $\lambda_1 > 0$ $\lambda_2 > 0$ $\lambda_3 > 0$

Kuhn Tucker conditions are:

$$y - \lambda_1 + \lambda_2 = 0$$

$$x - \lambda_1 + \lambda_3 = 0$$

$$x + y = 6$$

$$x = 0 \quad y = 0$$

From the last three conditions one contradiction arises

Two possible solutions

1) $x = 0$ and $y = 0$

2) $x = 3$ and $y = 3$

The second one produces the higher value of the objective function, then it is the solution of the problem

2) Using the modified lagrangean

$$M(x, y) = xy - \lambda_1(x + y - 6)$$

Kuhn-Tucker conditions for the modified Lagrangean:

$$\begin{array}{lll} x \geq 0, & y - \lambda_1 \leq 0 & x(y - \lambda_1) = 0 \\ y \geq 0 & x - \lambda_1 \leq 0 & y(x - \lambda_1) = 0 \\ \lambda_1 \geq 0, & x + y \leq 6, & \lambda_1(x + y - 6) = 0 \end{array}$$

Kuhn-Tucker conditions for the modified Lagrangean:

$$\begin{aligned}x &\geq 0, & y - \lambda_1 &\leq 0 & x(y - \lambda_1) &= 0 \\y &\geq 0 & x - \lambda_1 &\leq 0 & y(x - \lambda_1) &= 0 \\ \lambda_1 &\geq 0, & x + y &\leq 6, & \lambda_1(x + y - 6) &= 0\end{aligned}$$

Consider a case where $x=0$ and $y=0$, then:

$$\begin{aligned}-\lambda_1 &\leq 0 \\-\lambda_1 &\leq 0 \\ \lambda_1 &\geq 0, & x + y &\leq 6, & \lambda_1(x + y - 6) &= 0\end{aligned}$$

These conditions are satisfied only for $\lambda_1 = 0$

Then $x=0$ $y=0$ is a candidate to the solution (the derivatives of the objective function are equal to zero in this point)

Consider a case where $x>0$ and $y=0$, then:

Replacing these values in the first condition we get $\lambda_1 = 0$

Replacing $\lambda_1 = 0$ in the second condition we get $x \leq 0$

A contradiction with the initial assumption $x>0$.

Kuhn-Tucker conditions for the modified Lagrangean:

$$\begin{aligned}x &\geq 0, & y - \lambda_1 &\leq 0 & x(y - \lambda_1) &= 0 \\y &\geq 0 & x - \lambda_1 &\leq 0 & y(x - \lambda_1) &= 0 \\ \lambda_1 &\geq 0, & x + y &\leq 6, & \lambda_1(x + y - 6) &= 0\end{aligned}$$

Consider a case where $x=0$ and $y>0$, then:

Replacing these values in the second condition we get $\lambda_1 = 0$

Replacing $\lambda_1 = 0$ in the first condition we get $y \leq 0$

A contradiction with the initial assumption $y>0$.

Consider the case $x>0$ and $y>0$

$$\begin{aligned}y - \lambda_1 &= 0 \\x - \lambda_1 &= 0 \\ \lambda_1 &\geq 0, & x + y &\leq 6, & \lambda_1(x + y - 6) &= 0\end{aligned}$$

Then $y = x = \lambda_1 > 0$ then the last condition implies $x + y = 6$ and then $x=y=3$

As in the procedure of the KT condition