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Lecture 9 – Differential equations

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## Integration and differential equations.

- Definition: a differential equation includes the derivative of a function as a variable to model the change in the level of a variable
- E.g.

$$\frac{dy}{dx} = a \quad \text{or} \quad \frac{dy}{dx} = ax^2 \quad \text{or} \quad \frac{dy}{dx} = y - ax^2$$

- These examples have only first derivatives. They are called first order differential equations. Second order differential equations include second derivatives and so on.

$$\frac{d^2 y}{dx^2} = a + \frac{dy}{dx}$$

## Differential equations.

- Linear equations have no powers in x and no interaction terms between y and x
- Homogeneous equations are ones such that if we multiply all variables by a constant, the differential equation is unchanged.

$$\frac{dy}{dx} = x \quad \text{or} \quad a \frac{dy}{dx} = ax \quad \text{homogeneous}$$

$$\frac{dy}{dx} = x^2 \quad \text{or} \quad a \frac{dy}{dx} = (ax)^2 \quad \text{non-homogeneous}$$

- In the most common examples, x = time, usually indicated by 't'
- In economics, decisions are rarely taken continuously. Hence differential equations are usually approximations to processes that might be more exactly represented via difference equations:

$$\frac{dy}{dt} = ay \quad \text{or} \quad y_t - y_{t-1} = ay_{t-1}$$

- To solve a differential equation completely we also need information about y at some time t. E.g. starting point or end point.

## Solving a simple homogeneous equation

- We are interested in finding the value of an underlying variable at any point in time when we are only given information on how the variable evolves over time (the differential)

- Example 1

$$\frac{dx}{dt} = f(t) \equiv \frac{dx}{dt} - f(t) = 0$$

- Integrate both sides wrt t

$$\int \frac{dx}{dt} dt = \int f(t) dt$$

$$\Rightarrow x = \int f(t) + c$$

- Eg.  $f(t) = t^2 - 1 = dx/dt$
- Gives  $x = t^3 / 3 - t + c$
- So given t can work out the value of x in any period

## Solving a simple homogeneous equation.

$$\frac{dy}{dt} = ay$$

- One way to approach the solution is to treat  $dy$  and  $dt$  as separate items and integrate:

$$\frac{1}{y} dy = a dt$$

$$\int \frac{1}{y} dy = \int a dt$$

$$\ln(y) = at + c$$

$$y = e^c e^{at} = Ae^{at}$$

## Solving a simple homogeneous equation.

- Now  $y(t) = Ae^{at}$
- (is an equation commonly used to describe the evolution of GDP over time where  $a$  = rate of growth)
- Note that  $A$  could be any value
- (in the growth literature  $A$  is used as a measure of technical progress)
- $y(t) = Ae^{at}$  is said to be the **general solution** to the differential equation

$$\frac{dy}{dt} = ay$$

- For a particular value of  $A$  this becomes a **particular solution**
- If we start off the process  $y(0)=A$ , then this gives the **definite solution**
- Note the solution ( $y(t) = Ae^{at}$ ) is free of any derivative and is not a numerical value rather a function (or a “time path”) giving the value of  $y$  at any point in time

## Solving a simple non-homogeneous equation

$$\frac{dy}{dt} = ay + b$$

- The solution to the related homogeneous equation is called the **complementary function** ( $y_c$ )
- We call the **particular integral** ( $y_p$ ) any particular solution to the non-homogeneous equation)
- The sum of the complementary function and the particular integral constitutes the **general solution** of a 1<sup>st</sup> order linear non-homogenous differential equation

## Solving a simple non-homogeneous equation.

$$(1) \quad \frac{dy}{dt} = ay + b$$

To solve proceed in 2 steps:

Look for **any** value that satisfies the equation  
the simplest is to let  $y(t) = k =$  a constant  
then  $dy/dt = 0$  and (1) becomes

$$- ay = b$$

and

$$y(t) = -b/a = k \quad (a \neq 0)$$

This solution to the non-homogeneous equation is called the particular integral



## Solving a simple non-homogeneous equation.

Step 2:

We consider the homogeneous related equation

$$\frac{dy}{dt} = ay$$

And we have already seen that the solution to this type of homogenous differential equation is  $y(t) = Ae^{at}$

## Solving a simple non-homogeneous equation

- The solution to the related homogeneous equation (**complementary function** ( $y_c$ )) in this case is

$$y(t) = Ae^{at}$$

- A particular solution to the non-homogeneous equation (**particular integral** ( $y_p$ )) in this case is

$$y(t) = -b/a$$

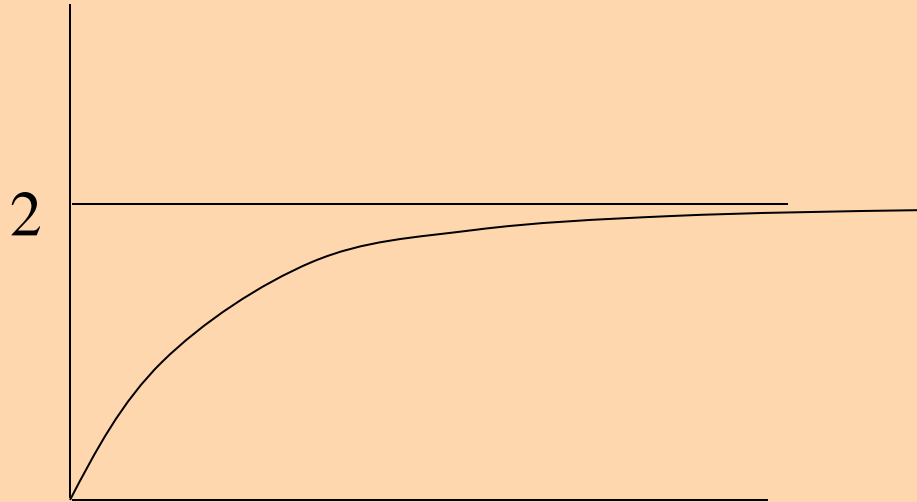
- The sum of the complementary function and the particular integral constitutes the **general solution** of a 1<sup>st</sup> order linear non-homogeneous differential equation, in this case:

$$y(t) = y_c + y_p = Ae^{at} - b/a$$

- to get the definite solution we need to impose an initial condition:
- the value of  $y$  at  $t = 0$
- In this example  $y(0) = Ae^0 - b/a = A - (b/a)$ ,
- $A = y(0) + (b/a)$
- $y(t) = (y(0) + (b/a))e^{at} - (b/a)$

## Example: $a=-1$ , $b =2=-K$

$$y = 2 - 2e^{-t}$$



- Often in economics the particular integral will represent the equilibrium of a system (here  $y = 2$ ), while the complementary function supplies the dynamics to get there (from the initial condition value)

# Solving a simple non-homogeneous equation

## Example 2

Solve  $dy/dt + 2y(t) = 6$  with an initial condition  $y(0) = 10$

The constant solution  $dy/dt = 0$  gives the particular integral

$$y_p = 6/2 = 3$$

The complementary function is the solution to  $dy/dt = -2y(t)$   
(with the constant set to zero)

- Or:

$$\frac{1}{y} \frac{dy}{dt} = -2$$

$$\int \frac{1}{y} \frac{dy}{dt} dt = \int -2 dt$$

- Integrating both sides wrt  $t$ :

$$\ln(y) + c_1 = -2t + c_2$$

$$\ln(y) = -2t + (c_2 - c_1)$$

$$y = e^{-2t} e^k; k = c_2 - c_1$$

$$y = Ae^{-2t}$$

## Solving a simple non-homogeneous equation

So the general solution is:

$$y(t) = y_c + y_p = Ae^{-2t} + 3$$

The definite solution is to use the value for the initial conditions  $t=0$ ,  $y(0) = 10$

$$y(0) = Ae^0 + 3 = A + 3 \text{ so then } A = y(0) - 3 = 10 - 3 = 7$$

So then

$$y(t) = [10 - 3]e^{-2t} + 3 = 7e^{-2t} + 3$$

This is saying the system converges to the value 3, starting from an initial value of 10 as  $t \rightarrow \infty$

# Solving a simple non-homogeneous equation

- Sometimes solution is written down as

$$\frac{dy}{dt} + vy = z$$

$$\Rightarrow y(t) = e^{-\int v dt} \left( A + \int z e^{\int v dt} dt \right)$$

- where A is an arbitrary constant

$e^{-\int v dt} A$  is the complementary function

$e^{-\int v dt} \int z e^{\int v dt}$  is the particular integral

- The particular integral  $y_p$  gives the inter-temporal equilibrium level of  $y(t)$
- The complementary function  $y_c$  gives the deviation from the equilibrium
- For  $y(t)$  to be dynamically stable  $y_c$  must approach 0 as  $t$  goes to infinity

## Solving a simple non-homogeneous equation

- Eg: Find the general solution for  $dy/dt + 4y = 12$

- Let  $v = 4$  and  $z = 12$

- Using

$$y(t) = e^{-\int v dt} \left( A + \int z e^{\int v dt} dt \right)$$

$$\Rightarrow y(t) = e^{-4t} \left( A + \int 12 e^{4t} dt \right) = e^{-4t} \left( A + 3e^{4t} \right)$$

$$\Rightarrow y(t) = Ae^{-4t} + 3$$

- The 1<sup>st</sup> term is the complementary function and the second is the particular integral
- As  $t$  goes to infinity the complementary function goes to zero and so  $y(t)$  approaches the level of the particular integral
- Often helpful to check the solution by differentiating backwards

# Exact differential equations

a total differential of  $F(y,t)$  is:

$$dF = \frac{\partial F}{\partial y} dy + dt \frac{\partial F}{\partial t}$$

- If  $dF = 0$ , the result is called an exact differential equation. The general solution should be of the form  $F = c$  ( $c$  is a constant)
- Usually though we seek to find  $F$  given the differential.
- The statement that a differential equation of the form:

$$0 = Mdy + dtN$$

( $M$  and  $N$  are functions)

is exact is equivalent to the statement that there exists  $F$  such that

$$M = \frac{\partial F}{\partial y}; N = \frac{\partial F}{\partial t}$$

- By Young's theorem: 
$$\frac{\partial^2 F}{\partial y \partial t} = \frac{\partial^2 F}{\partial t \partial y}$$

- So  $F$  exists provided: 
$$\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$$



## Solving exact differential equations

- Step 1. Check that  $\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$  If it is then we have an exact equation.

- If it is not exact then we may still be able to solve (see later)

- Step 2. Integrate M partially with respect to y. The result is:

$$F = \int M dy + g(t)$$

- Note the term g(t) which we do not know yet.
- Step 3. Partially differentiate the result in 2 with respect to t to get N:

$$\frac{\partial F}{\partial t} = \frac{\partial \int M dy}{\partial t} + g'(t) = N$$

- Note that g' is the derivative of g with respect to t.
- Step 4. We know N so we can solve this equation to find g'. From that we can integrate g' to find g then use starting conditions to find the general solution.

## Solving exact differential equations - example

- Solve  $\frac{dy}{dt} = 4t$
- Step 1.
- We rewrite the differential equation:  $0 = dy - 4tdt$
- So  $M = 1$  and  $N = -4t$ .  $\frac{\partial M}{\partial t} = 0 = \frac{\partial N}{\partial y}$
- Step 2. Integrate  $M$  partially with respect to  $y$ . The result is:  
$$F = \int M dy + g(t) = y + g(t)$$
- Step 3. Partially differentiate the result in 2 with respect to  $t$  to get  $N$ :  
$$\frac{\partial F}{\partial t} = \frac{\partial y}{\partial t} + g'(t) = 0 + g'(t) = -4t$$
- Step 4. find  $g'$ .  $g' = -4t$ , so integrate  $g'$  to find  $g$ :  $\int g' dt = k - 2t^2$
- So,  $F = k + y - 2t^2$  or  $c = y - 2t^2$  or  $y = 2t^2 + c$

## Integrating factors

- Sometimes M and N mean that the differential equation is inexact, but we can still find another function of t and y which we can multiply through by to get an exact equation.

- Example.  $0 = tdy + 2ytdt$        $\frac{\partial M}{\partial t} = 1 \neq 2 = \frac{\partial N}{\partial y}$

- Multiply through by t to get:  $0 = t^2 dy + 2ytdt$

- Step 1. Check  $\frac{\partial M}{\partial t} = 2t = \frac{\partial N}{\partial y}$

- Step 2. Integrate M partially with respect to y. The result is:

$$F = \int M dy + g(t) = yt^2 + g(t)$$

- Step 3. Partially differentiate the result in 2 with respect to t to get N:

$$\frac{\partial F}{\partial t} = \frac{\partial \int M dy}{\partial t} + g'(t) = 2ty + g' = N = 2ty$$

- Step 4. From this we get  $g' = 0$  or  $g = k$ ,

$$F = yt^2 + k \quad \text{or} \quad c = yt^2 \quad \text{or} \quad y = \frac{c}{t^2}$$