Elections and the Representation of Preferences over Infinite Sets

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Abstract

Voting theory has always focused on mechanism design, but this paper shows that voting theory is also a useful tool in the field of preference representation. Both the lexicographic order on $\Re^n$ and the threshold of detectable difference relation are pairwise majority voting aggregates of utility functions. Pareto dominance on $\Re^n$ and the threshold of detectable difference relation are pairwise unanimous voting aggregates of utility functions. Separability conditions are established for voting aggregates, and used to show that the lexicographic order is not a pairwise unanimous voting aggregate of utility functions, and Pareto dominance is not a pairwise majority voting aggregate of utility functions.

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1. Introduction

Every preference relation on a finite set is representable as a pairwise majority voting aggregate of linear orders [Saari, 1994]. In other words, given even an unreasonable preference relation P on a finite set X (for instance, P might contain preference cycles such as $xPyPzPx$) one can find a finite collection of voters, each with linear preferences over X, such that if an election is held between alternatives x and y in X, x wins a majority of the voters if and only if xPy. It is also the case that every *transitive* preference relation on a finite set is representable as a pairwise *unanimous* voting aggregate of linear orders [Knoblauch, 1999].

These facts have two interpretations, one with a negative flavor, the other positive. The first is that both majority voting and unanimous voting lead to troublesome outcomes. For example, pairwise majority voting by voters with linear preferences may generate preference cycles, and pairwise unanimous voting by voters with linear preferences may generate intransitive indifference. These are called voting paradoxes, so the first interpretation of the two results cited above may be restated as follows: majority voting is subject to every imaginable paradox and unanimous voting is subject to every imaginable paradox involving transitive outcomes.

The second interpretation is that preferences over a finite set not representable by a utility function are representable by an election. Representations for preferences are important tools. They provide conceptual handles for preference relations. They also provide more concrete services, such as helping to answer the following question, which is of obvious economic interest: for a given preference relation, which subsets of alternatives contain maximally preferred elements? For example, if a preference relation is represented by a continuous utility function, then every compact subset of alternatives contains a maximally preferred element. Similarly, every compact subset contains a maximally preferred element when the preference relation in question is represented by an upper semi-continuous weak utility function [Subiza and Peris, 1997] or by two continuous functions [Chateauneuf, 1987].

This paper is dedicated to the study of election representations of preference relations not representable by utility functions. Since the preference relations not representable by utility functions that are of most interest to economists are preference relations on infinite sets–for example, Pareto dominance on $\Re^n$, the lexicographic order on $\Re^n$ and the threshold of detectable difference on $\Re$–this paper will concern itself with elections with infinitely many candidates. It would be unreasonable to hope (or fear if one adopts the mechanism design interpretation of voting theory, rather than the preference representation interpretation)–that every every preference relation on an *infinite* set is representable as a pairwise majority voting aggregate of utility functions, or that every transitive preference relation is representable as a pairwise unanimous voting aggregate of utility functions. In fact it will be shown below that a pairwise unanimous voting aggregate of utility functions is weakly separable. It will follow that the lexicographic order on $\Re^n$, $n \geq 2$, is not a pairwise unanimous voting aggregate. Also, any pairwise majority voting aggregate of utility functions will be shown to satisfy a separability condition. It will follow that Pareto dominance on $\Re^n$, $n \geq 3$, is not a pairwise majority voting aggregate.

Notice that the voters in this paper are assumed to have preferences generated by utility functions. It is desirable to endow voters with familiar, simple, highly structured preferences, since voting theorists interested in mechanism design want to see if very rational voters produce reasonable aggregate preferences, and voting theorists interested in preference representation want familiar, simple, highly structured representations. in the finite alternative set context, voters are most often assumed to have linear preferences, which are familiar, simple, and more structured than utility based preferences (since over a finite set, every linear order is generated by a utility function, and is in addition complete). On the other hand, in the context of infinite alternative sets, voters are best assumed to be guided by utility functions, since preferences generated by utility functions are more often encountered by economists, and are simpler and more highly structured (they are not in general complete, but linear orders are not in general separable and therefore include exotic examples such as well orderings of the reals).

The results of this paper mentioned in the previous paragraph but one indicate that there are limits on the use of elections to represent preference relations. On a more positive note, it will be shown that Pareto dominance on $\Re^n$ is a pairwise unanimous voting aggregate of n utility functions, the lexicographic order on $\Re^n$ is a pairwise majority voting aggregate of $2^n - 1$ utility functions, and the threshold of detectable difference relation on $\Re$ ($x \succ_T y$ if $x \geq y + 1$) is a pairwise majority voting aggregate of four utility functions as well as a pairwise unanimous voting aggregate of three utility functions.

The paper is organized as follows. Section 2 contains preliminaries. Section 3 establishes weak separability of unanimous voting aggregates and discusses the three best-known examples of preference relations not represented by utility functions: lexicographic order, Pareto dominance and the threshold of detectable difference relation. Section 4 establishes a separability condition for majority voting aggregates and revisits the three examples. Section 5 consist of concluding remarks.

2. Preliminaries

A *binary relation* on a set X is a collection of ordered pairs of elements of X. A binary relation B on X is *asymmetric* if $(x, y) \in B$ implies $(y, x) \notin B$.

A *preference relation* $\succ$ on X is an asymmetric binary relation on X. It will be convenient to write $x \succ y$ in place of $(x, y) \in \succ$.

If $\succ$ is a preference relation on X, the associated *weak preference relation* on X is defined by $x \succsim y$ if not$(y \succ x)$, and the associated *indifference relation* on X is defined by $x \sim y$ if $x \succsim y$ and $y \succsim x$.

A preference relation $\succ$ on X is *transitive* if $x \succ y \succ z$ implies $x \succ z$ for all $x, y, z \in X$; *complete* if $x \sim y$ implies $x = y$ for all $x, y \in X$.

A *linear order* on X is a transitive, complete preference relation.

Suppose $\succ_1, \succ_2, \ldots, \succ_m$ are preference relations on a finite set X. The *pairwise majority voting aggregate* $\succ_{MV(\succ_1, \succ_2, \ldots, \succ_m)}$ –or simply $\succ_{MV}$ when the context is clear–is the preference relation on X defined by

$$x \succ_{MV} y \text{ if } |\{i\colon\ x \succ_i y\}| > |\{i\colon\ y \succ_i x\}|.$$

where $|\{\ .\ \}|$ denotes the number of elements of $\{\ .\ \}$. The *pairwise unanimous voting aggregate* $\succ_{UV(\succ_1, \succ_2, \ldots, \succ_m)}$ is the preference relation on X defined by $x \succ_{UV} y$ if $x \succsim_i y$ for all i, $1 \leq i \leq m$, and $x \succ_i y$ for some i, $1 \leq i \leq m$.

A *utility function* representing preference relation $\succ$ on X is a function u: $X \to \Re$ such that, for all $x, y \in X$, $x \succ y$ if and only if $u(x) > u(y)$.

The *Pareto dominance relation* $\succ_{PD}$ on $\Re^n$ is defined by $x \succ_{PD} y$ if $x_i \geq y_i$ for all i, $1 \leq i \leq n$, and $x_i > y_i$ for some i, $1 \leq i \leq n$; the *lexicographic order* $\succ_L$ on $\Re^n$ is defined by $x \succ_L y$ if there is a positive integer $k \leq n$ such that $x_i = y_i$ for $i < k$ and $x_k > y_k$; the *threshold of detectable difference* relation $\succ_T$ on $\Re^+$ is defined by $x \succ_T y$ if $x \geq y + 1$.

A set $A \subseteq X$ is *dense* in preference relation $\succ$ on X if $x \succ y$ implies $x \succ a \succsim y$ for some $a \in A$. A preference relation $\succ$ on X is *separable* if there is a countable $A \subseteq X$ such that A is dense in $\succ$.

A set $A \subseteq X$ is *weakly dense* in preference relation $\succ$ on X if $x \succ y$ implies $x \succsim a \succsim y$ for some $a \in A$. A preference relation $\succ$ on X is *weakly separable* if there is a countable $A \subseteq X$ such that A is weakly dense in $\succ$.

3. *Unanimous Voting.*

Of course Pareto dominance on $\Re^n$ can be represented as a pairwise unanimous voting aggregate over $\Re^n$: for $1 \leq i \leq n$ define u^i: $\Re^n \to \Re$ by $u^i(x) = x_i$ and define $\succ_i$ to be the preference relation on $\Re^n$ represented by the utility function u^i. Then for $x, y \in \Re^n$,

$$\begin{aligned} x \succ_{UV} y & \text{ if and only if } x \succsim_i y \text{ for all } i \text{ and } x \succ_i y \text{ for some } i \\ & \text{ if and only if } x_i \geq y_i \text{ for all } i \text{ and } x_i > y_i \text{ for some } i \\ & \text{ if and only if } x \succ_{PD} y. \end{aligned}$$

However, the lexicographic order on $\Re^n$, $n \geq 2$, is not a pairwise unanimous voting aggregate, as will be shown using the following proposition.

Proposition 1. *Every pairwise unanimous voting aggregate of preference relations generated by utility functions is weakly separable.*

Corollary 1. *The lexicographic order $\succ_L$ on $\Re^n$, $n \geq 2$, is not a pairwise unanimous voting aggregate of preference relations generated by utility functions.*

Proof of Corollary 1. Suppose $A \subseteq \Re^n$ is weakly dense in $\succ_L$. For r real, $(r, 2, 2, \ldots, 2) \succ_L (r, 1, 1, \ldots, 1)$, so that there exits $a \in A$ such that $(r, 2, 2, \ldots, 2) \succsim_L a \succsim_L (r, 1, 1, \ldots, 1)$. Then $a_1 = r$. In other words, for each real r, there exists $a \in A$ such that $a_1 = r$. Therefore A is uncountable. Since every subset of $\Re^n$ that is weakly dense in $\succ_L$ is uncountable, $\succ_L$ is not weakly separable. By Proposition 1, $\succ_L$ is not a pairwise unanimous voting aggregate of preference relations generated by utility functions.

Proof of Proposition 1.

Suppose preference relations $\succ_1, \succ_2, \ldots, \succ_m$ on X are represented by utility functions $u^1, u^2, \ldots, u^m$ respectively. Then each $\succ_i$ is separable by Debreu [1964].

If $m = 1$, $\succ_{UV} = \succ_1$ which is separable and therefore weakly separable.

If $m > 1$, fix $i \leq m$ and choose $A^i \subseteq X$ such that A^i is countable and dense in $\succ_i$. Fix $a \in A^i$. If there exists $b \in X$ such that $u^i(b) = u^i(a)$ and $u^j(b) = \sup\{u^j(x)\colon\ u^i(x) = u^i(a)\}$ for all $j \neq i$, $1 \leq j \leq m$, then choose one such b and let $A(a, j) = \{b\}$ for all $j \neq i$, $1 \leq j \leq m$. If no such b exists, let $A(a, j) = \{b^{j,1}, b^{j,2}, \ldots\}$ chosen so that $u^i(b^{j,l}) = u^i(a)$ and

$$u^j(b^{j,l}) > \begin{cases} \sup\{u^j(x)\colon\ u^i(x) = u^i(a)\} - \frac{1}{l} & \text{if the sup is finite.} \\ l & \text{otherwise.} \end{cases}$$

Now let

$$A^{i\prime} = \bigcup_{\substack{a \in A^i \\ j \neq i \\ 1 \leq j \leq m}} A(a, j).$$

Let

$$A = \bigcup_{1 \leq i \leq m} A^{i\prime}.$$

That A is countable follows from the countability of each A^i and each $A(a, j)$.

To prove that A is weakly dense in $\succ_{UV}$, suppose $x \succ_{UV} y$. Then $x \succ_i y$ for some i. Fix $a \in A^i$ such that $x \succ_i a \succsim_i y$. Then $x \succ_i b \succsim_i y$ for all $b \in A(a, j)$, $j \neq i$. If follows that $x \succsim_{UV} b$ for all $b \in A(a, j)$, $j \neq i$. It remains to find b and j such that $j \neq i$, $b \in A(a, j)$ and $b \succsim_{UV} y$.

Case 1. $u^i(a) > u^i(y)$. Then $a \succ_i y$, $b \succ_i y$ for all $b \in A(a,j)$, $j \neq i$ and $b \succsim_{UV} y$ for all $b \in A(a,j)$, $j \neq i$.

Case 2. $u^i(y) = u^i(a)$, $u^j(y) = \sup\{u^j\colon\ u^i(x) = u^i(a)\}$ for all $j \neq i$. Then there is a $b \in X$ such that $A(a,j) = \{b\}$ for all $j \neq i$. Therefore $b \succsim_j y$ for $j \neq i$ so that $b \succsim_{UV} y$.

Case 3. $u^i(y) = u^i(a)$ and there exists $j \neq i$ such that $u^j(y) < \sup\{u^j(x)\colon\ u^i(x) = u^i(a)\}$. For this j, there exists $b^{j,l} \in A(a,j)$ such that $u^j(b^{j,l}) > u^j(y)$. Then $b^{j,l} \succ_{UV} y$ ∎

The following example ends the section on a positive note for those interested in elections as tools for preference representation, a note of warning for those who hope that Proposition 1 is an indication that allowing infinite alternative sets introduces no new voting paradoxes.

Example 1. The threshold of detectable difference relation $\succ_T$ is transitive, but not representable by a utility function, since, for example, $1 \succsim_T \frac{7}{4} \succsim_T \frac{5}{2}$, but $\frac{5}{2} \succ_T 1$. To obtain $\succ_T$ as a pairwise unanimous voting aggregate of three preference relations governed by utility functions, first define functions e, o, t, u, v on $\Re^+$ as follows. Let $e(x)$ be the number of positive even integers that are less than or equal to x. Let $o(x)$ be the number of positive odd integers that are less than or equal to x.

$$\begin{aligned} \text{Let } t(x) &= x - e(x) \\ u(x) &= x - o(x) \\ v(x) &= 2e(x) + 2o(x) - x \end{aligned}$$

Let $x \succ_1 y$ if $t(x) > t(y)$. In other words, $\succ_1$ is the preference relation on $\Re^+$ represented by utility function t. Similarly, let $\succ_2$ and $\succ_3$ be the preference relations represented by utility functions u and v respectively.

To see that $\succ_T = \succ_{UV(\succ_1,\succ_2,\succ_3)}$, first suppose $x > y + 1$. If $e(x) = e(y)$ or $e(x) = e(y) + 1$, then

$$t(x) = x - e(x) > y + 1 - (e(y) + 1) = t(y).$$

If $e(x) \geq e(y) + 2$, then

$$t(x) - t(y) = (e(x) - e(y)) + (x - 2e(x)) - (y - 2e(y))$$
$$> 2 + 0 - 2 = 0$$

Similarly, if $o(x) = o(y)$ or $o(x) = o(y) + 1$, then

$$u(x) = x - o(x) > y + 1 - (o(y) + 1) = u(y).$$

If $o(x) \geq o(y) + 2$, then

$$u(x) - u(y) = o(x) - o(y) + (x - 2o(x)) - (y - 2o(y))$$
$$> 2 - 1 - 1 = 0.$$

Also, $-1 < e(x) + o(x) - x \leq 0$ so that

$$v(x) = (e(x) + o(x)) + (e(x) + o(x) - x) > (e(x) + o(x)) - 1$$
$$\geq e(y) + o(y) \text{ (since } x > y + 1)$$
$$\geq (e(y) + o(y)) + (e(y) + o(y) - y) = v(y).$$

In summary, if $x > y + 1$ then $x \succ_{UV} y$.

Next, suppose $x = y+1$. Then either $e(x) = e(y)$ and $o(x) = o(y)+1$ or $e(x) = e(y)+1$ and $o(x) = o(y)$. In both cases, $t(x) \geq t(y)$, $u(x) \geq u(y)$ and $v(x) > v(y)$.

In summary, if $x = y + 1$, then $x \succ_{UV} y$.

Next, suppose $y < x < y + 1$. Then either $e(x) = e(y)$ and $o(x) = o(y) + 1$, or $e(x) = e(y) + 1$ and $o(x) = o(y)$, or $e(x) = e(y)$ and $o(x) = o(y)$. In the first case, $t(x) > t(y)$ and $u(x) < u(y)$. In the second case, $t(x) < t(y)$ and $u(x) > u(y)$. In the third case, $t(x) > t(y)$ and $v(x) < v(y)$.

In summary, if $y < x < y + 1$ then $x \sim_{UV} y$.

If $x = y$, clearly $x \sim_{UV} y$.

Next, if $y - 1 < x < y$, then $x < y < x + 1$ and by the previous paragraph but one, $x \sim_{UV} y$. Also, if $x = y - 1$ then $y = x + 1$ and by the previous paragraph but three, $y \succ_{UV} x$. Finally, if $x < y - 1$, then $y > x + 1$ and by the previous paragraph but five, $y \succ_{UV} x$.

Combining the seven cases, $x \succ_{UV} y$ if and only if $x \geq y + 1$ if and only if $x \succ_T y$.

4. Majority Voting

The lexicographic order on $\Re^n$ is a pairwise majority voting aggregate of preference relations generated by utility functions: for positive integer $k \leq n$, let $u^j(x) = x_k$ if $2^{n-k} \leq j < 2^{n-k+1}$ (for example, if $n = 3$ there are four functions of the form $u(x) = x_1$, two of the form $u(x) = x_2$ and one of the form $u(x) = x_3$). For $1 \leq j < 2^n$, let $\succ_j$ be the preference relation on $\Re^n$ represented by u^j. Now suppose $x, y \in \Re^n$ and $x \succ_L y$. Then there exists positive integer $k \leq n$ such that $x_i = y_i$ if $i < k$ and $x_k > y_k$. Therefore, $u^j(x) > u^j(y)$ if $2^{n-k} \leq j < 2^{n-k+1}$ and $u^j(x) = u^j(y)$ if $2^{n-k+1} \leq j < 2^n$. It follows that $|\{j\colon x \succ_j y\}| \geq |\{j\colon 2^{n-k} \leq j < 2^{n-k+1}\}| = 2^{n-k+1} - 2^{n-k} = 2^{n-k}$, while $|\{j\colon y \succ_j x\}| \leq |\{j\colon 1 \leq j < 2^{n-k}\}| = 2^{n-k} - 1$. In other words, if $x \succ_L y$, then $x \succ_{MV} y$. Also, if $x \sim_L y$, then $x = y$ so that $x \sim_{MV} y$. From these two implications, it follows that $\succ_L = \succ_{MV(\succ_1, \succ_2, \ldots, \succ_{2^n - 1})}$.

On the other hand, Pareto dominance on $\Re^n$, $n \geq 3$, is not a pairwise majority voting aggregate, as the following proposition will show.

Proposition 2. *Suppose that*

1) $\succ_{MV}$ on X is a pairwise majority voting aggregate of preference relations generated by utility functions.

2) $X = \{A^k\colon k \in K\} \cup \{b^k\colon k \in K\}$ where the A^k's are distinct, the b^k's are distinct and $A^j \neq b^k$ for all $j, k \in K$.

3) $\succ_{MV}$ is completely given by $A^k \succ_{MV} b^k$ for all $k \in K$.

Then K is countable.

Corollary 2. *Pareto dominance on $\Re^n$, $n \geq 3$, is not a pairwise majority voting aggregate of preference relations generated by utility functions.*

Proof of Corollary 2. Set $A^r = (r+2, r, 1-r, 0, \ldots, 0)$ and $b^r = (r, r, 1-r, 0, \ldots, 0)$ for $r \in [0,1]$. Let $X = \{A^r \colon r \in [0,1]\} \cup \{b^r \colon r \in [0,1]\}$. Then $X \subseteq \Re^n$ and $\succ_{PD}$ on X satisfy hypotheses 2) and 3) of Proposition 2. Since $[0,1]$ is uncountable, $\succ_{PD}$ on X does not satisfy hypothesis 1) and therefore $\succ_{PD}$ on $\Re^n$ is not a pairwise majority voting aggregate of preference relations generated by utility functions. ∎

Proof of Proposition 2. Suppose X and $\succ_{MV}$ are as in the statement of the proposition, so that $\succ_{MV} = \succ_{MV(\succ_1, \succ_2, \ldots, \succ_m)}$ where preference relations $\succ_1, \succ_2, \ldots, \succ_m$ on X are represented by utility functions $u^1, u^2, \ldots, u^m$ respectively.

Lemma 1. *If K is uncountable, then there exists uncountable $C \subseteq K$ such that, for all $j, k \in C$ and $1 \leq i \leq m$,*

$$u^i(A^k) > u^i(b^k) \text{ if and only if } u^i(A^j) > u^i(b^j)$$
$$u^i(A^k) < u^i(b^k) \text{ if and only if } u^i(A^j) < u^i(b^j).$$

Proof. Define $f \colon K \to P(M) \times P(M)$, where $P(M)$ is the power set on $M = \{1, 2, \ldots, m\}$, by

$$f(k) = (\{i \colon u^i(A^k) > u^i(b^k)\}, \{i \colon u^i(A^k) < u^i(b^k)\}).$$

Since $P(M) \times P(M)$ is a finite set and K is uncountable, there exist $M^1, M^2 \subseteq M$ and uncountable $C \subseteq K$ such that $f(k) = (M^1, M^2)$ for all $k \in C$. Then $f(j) = f(k)$ for all $j, k \in C$ ∎

Lemma 2. *Suppose $K = \{1, 2\}$ and for all i, $1 \leq i \leq m$,*

$$u^i(A^1) > u^i(b^1) \text{ if and only if } u^i(A^2) > u^i(b^2)$$
$$u^i(A^1) < u^i(b^1) \text{ if and only if } u^i(A^2) < u^i(b^2)$$

Then there exists i such that $u^i(A^1) \geq u^i(b^1) \geq u^i(A^2) \geq u^i(b^2)$.

Proof. Since $A^1 \succ_{MV} b^1 \succsim_{MV} A^2$,

$$|\{i \colon u^i(A^1) \geq u^i(b^1)\}| > \frac{m}{2} \text{ and } |\{i \colon u^i(b^1) > u^i(A^2)\}| \geq \frac{m}{2}.$$

Therefore there is an i such that $u^i(A^1) \geq u^i(b^1) \geq u^i(A^2)$. Since $u^i(A^1) \geq u^i(b^1)$, by hypothesis $u^i(A^2) \geq u^i(b^2)$ ∎

Lemma 3. *Under the hypotheses of Lemma 2, there exist i such that*

$$u^i(A^1) > u^i(b^1) \geq u^i(A^2) > u^i(b^2)$$
$$\text{or} \quad u^i(A^1) = u^i(b^1) > u^i(A^2) = u^i(b^2).$$

Proof. The proof proceeds by induction on m.

If $m = 1$, since $A^1 \succ_{MV} b^1 \succsim_{MV} A^2 \succ_{MV} b^2$,

$$u^1(A^1) > u^1(b^1) \geq u^1(A^2) > u^1(b^2).$$

Now suppose $l > 1$ is a positive integer such that the lemma holds when $m < l$. It will be shown that the lemma holds when $m = l$.

If $m = l$, by Lemma 2 there exists i^o, $1 \leq i^0 \leq l$, such that

$$u^{i^o}(A^1) \geq u^{i^0}(b^1) \geq u^{i^0}(A^2) \geq u^{i^0}(b^2).$$

Either the conclusion of Lemma 3 holds for $i = i^0$ or

$$u^{i^o}(A^1) = u^{i^0}(b^1) = u^{i^0}(A^2) = u^{i^0}(b^2)$$

in which case $\succ_{MV(\succ_1, \succ_2, \ldots, \succ_{i^0-1}, \succ_{i^0+1}, \ldots, \succ_l)}$ and X satisfy the hypotheses of the proposition (since voter i^0 has no preferences, every pairwise election without her has the same outcome that it had with her) and the additional hypotheses of Lemma 3 (since the hypotheses of Lemma 2 hold for each i, $1 \leq i \leq l$, they hold for each $i \neq i^o$, $1 \leq i \leq l$). Also, $|\{1, 2, \ldots, l\} - \{i^0\}| = l - 1$ so that by the induction hypothesis, the conclusion of Lemma 3 holds for some $i \in \{1, 2, \ldots, l\} - \{i^o\}$ ∎

Lemma 4. *Under the hypotheses of Lemma 2, there is an i, $1 \leq i \leq m$, such that*

$$u^i(A^1) > u^i(b^1),\ u^i(A^2) > u^i(b^2) \text{ and either}$$
$$u^i(b^1) \geq u^i(A^2) \text{ or } u^i(b^2) \geq u^i(A^1).$$

Proof. The proof proceeds by induction on m. If $m = 1$, since $A^1 \succ_{MV} b^1 \succsim_{MV} A^2 \succ_{MV} b^2$,

$$u^1(A^1) > u^1(b^1) \geq u^1(A^2) > u^1(b^2).$$

Suppose $l > 1$ is a positive integer such that the lemma holds when $m < l$. It will be shown that the lemma holds when $m = l$.

If $m = l$, by Lemma 3 there are two integers in $\{1, 2, \ldots, m\}$ which for ease of notation we will assume are 1 and 2 such that both

$$u^1(A^1) > u^1(b^1) \geq u^1(A^2) > u^1(b^2) \text{ or } u^1(A^1) = u^1(b^1) > u^1(A^2) = u^1(b^2)$$

and

$$u^2(A^2) > u^2(b^2) \geq u^2(A^1) > u^2(b^1) \text{ or } u^2(A^2) = u^2(b^2) > u^2(A^1) = u^2(b^1).$$

Then either the conclusion of this lemma holds or

$$u^1(A^1) = u^1(b^1) > u^1(A^2) = u^i(b^2) \text{ and } u^2(A^2) = u^2(b^2) > u^2(A^1) = u^2(b^1)$$

in which case X and $\succ_{MV(\succ_3, \succ_4, \ldots, \succ_l)}$ satisfy the hypotheses of Proposition 2 (since voters 1 and 2 cancel each other out, every pairwise election has the same outcome without them that it had with them) and the hypothesis of Lemma 4 (since the hypotheses of Lemma 2 hold for each $i \in \{1, 2, \ldots, l\}$ they hold for each $i \in \{3, 4, \ldots, l\}$). Also, $|\{3, 4, \ldots, l\}| = l - 2$ so that by the induction hypothesis, the conclusion of Lemma 4 holds for some $i \in \{3, 4, \ldots, l\}$ ∎

Lemma 5. *Suppose that for every pair $j, k \in K$ with $j \neq k$ there is an i, $1 \leq i \leq m$, such that*

$$u^i(A^k) > u^i(b^k), \;\; u^i(A^j) > u^i(b^j) \text{ and either}$$
$$u^i(b^k) \geq u^i(A^j) \text{ or } u^i(b^j) \geq u^i(A^k)$$

Then K is countable.

Proof. The proof proceeds by induction on m. If $m = 1$, suppose $j, k \in K$, $j \neq k$. Then the open intervals $(u^1(b^k), u^1(A^k))$ and $(u^1(b^j), u^1(A^j))$ are nonempty and disjoint by the hypotheses of this lemma. Therefore, K must be countable.

Suppose $l > 1$ is a positive integer such that the lemma is true for $m < l$. it will be shown that the lemma is true for $m = l$. Suppose K is uncountable and consider the following cases.

Case 1. $u^l(A^k) > u^l(b^k)$ for uncountably many k. Then there exits rational q such that $u^l(A^k) > q > u^l(b^k)$ for $k \in C \subseteq K$, C uncountable. If $j, k \in C$, it is neither the case that $u^l(b^k) \geq u^l(A^j)$ nor that $u^l(b^j) \geq u^l(A^k)$. Therefore the hypothesis of the lemma holds for $\succ_1, \succ_2, \ldots, \succ_{l-1}$ and $X' = \{A^k\colon k \in C\} \cup \{b^k\colon k \in C\}$. Hypothesis 3) of Proposition 2 might not be satisfied, but since it was not used in the proof of the case $m = 1$, the induction hypothesis can be invoked to conclude that C is countable. The assumption that K is uncountable has led to a contradiction, that C is countable and uncountable.

Case 2. $u^l(A^k) \leq u^l(b^k)$ for $k \in C \subseteq K$, C uncountable. Then the hypotheses of the lemma hold for $\succ_1, \succ_2, \ldots, \succ_{l-1}$ and $X' = \{A^k\colon k \in C\} \cup \{b^k\colon k \in C\}$. By the induction hypothesis, C is countable. The assumption that K is uncountable has again led to the contradiction that C is countable and uncountable ∎

To prove Proposition 2, suppose K is uncountable. By Lemma 1 there is an uncountable $C \subseteq K$ such that if $j, k \in C$, $j \neq k$, then $\succ_1, \succ_2, \ldots, \succ_m$ and $\{A^j, A^k, b^j, b^k\}$ satisfy the hypotheses of Lemma 2. By Lemma 4, $\succ_1, \succ_2, \ldots, \succ_m$ and $X' = \{A^k\colon k \in C\} \cup \{b^k\colon k \in C\}$ satisfy the hypotheses of Lemma 5. Therefore by Lemma 5, C is countable. The assumption that K is uncountable has led to a contradiction, that C is countable and uncountable ∎

Proposition 2 and Corollary 2 demonstrate that majority voting is not the universally applicable preference representation tool over infinite sets that it is over finite sets. Nevertheless, the following example indicates that majority voting has potential as a tool for representing preferences over infinite sets.

Example 2. The threshold of detectable difference relation $\succ_T$ on $\Re^+$ is a pairwise majority voting aggregate of four preference relations generated by utility functions. For $x \in \Re^+$ let $e(x)$ and $o(x)$ be, as in Example 1, respectively the number of positive even integers and positive odd integers less than or equal to x. Let

$$t(x) = x - e(x)$$

$$u(x) = x - o(x)$$

$$v(x) = 4e(x) - x$$

$$w(x) = 4o(x) - x$$

and let $\succ_1, \succ_2, \succ_3, \succ_4$ be preference relations on $\Re^+$ represented by utility functions t, u, v, w respectively. The proof that $\succ_{MV(\succ_1,\succ_2,\succ_3,\succ_4)} = \succ_T$ follows the proof in Example 1 case by case.

4. Concluding Remarks.

There are two questions that may have occurred to the reader. First, why bother to search for representations for preference relations like those in this paper, lexicographic order, Pareto dominance and the threshold of detectable difference relation, whose associated indifference relations are intransitive, when many economists argue that such relations are so irrational that they are unlikely to be encountered in the real world? One answer is that once a voting representation is found for a preference relation, it can be argued that that preference relation is reasonable; it arises from utility functions via a reasonable process, voting.

Second, why bother to search for representations for a preference relation like the threshold of detectable difference relation, which is easily understood in terms of its simple definition ($x \succ_T y$ if $x \geq y + 1$) and a simple scenario: I prefer the larger of two chocolate bars, but I can only detect a size difference if the bars differ in weight by at least one gram? The answer is that once a voting representation is found for $\succ_T$, it becomes apparent that the previous understanding--in terms of a simple definition and a simple scenario--was not as complete as one might have thought. A voting representation can reveal complex structure in a seemingly simple preference relation.

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