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Comparing Electoral Systems: A Geometric Analysis

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Abstract:

This paper constructs a game-theoretic model of elections in alternative electoral systems with three or four candidates. Each electoral system specifies how the platforms of the candidates and their scores give rise to an outcome. When geometrical analysis shows that two outcomes can compete against each other for victory, a pivot probability is associated to that pair. Each voter is rational and picks the candidate that maximizes her expected utility, which results from the balancing of her preferences and beliefs about the pivot-probabilities. Candidate positioning is endogenous and the result of a Nash game. The possible equilibria are computed for plurality and runoff majority systems.

Keywords: electoral system, outcome simplex, strategic voting, pivot probability, positional equilibrium, runoff system, median voter theorem.

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0. Introduction

This paper deals with the comparative analysis of positional equilibria under alternative electoral systems. Up to now, most electoral systems have been examined separately and in different frameworks. I propose a unified model of elections with sophisticated voting and endogenous platform positioning with three and four parties.[1]

I assume strategic Nash choice of platforms by candidates, as most of the literature with endogenous positioning.

The behavior of voters is a more controversial issue. Early literature considered sincere voting, but this assumption is unreasonable for many situations, and there is strong empirical evidence for sophisticated voting (Riker,1973)). Recent literature therefore examined sophisticated voting, and even strategic voting (Austen-Smith and Banks, 1988, Besley and Coate, 1997...). This paper takes an intermediate view, which was proposed by Myerson and Weber (1993): voters are rational and therefore able of sophisticated voting, but are not really able to interact with each other. Therefore, they take into account the available information regarding how strong the candidates are and try not to waste their vote on unlikely winners: they maximize their expected utility.

The literature regarding majority or plurality elections is quite extensive. Negative results have been emphasized: plurality elections can lead to the election of a less preferred candidate (see for example Fishburn (1986), Wright and Riker (1989)).

[1] The electoral system has no impact when there are two candidates. They position at the median of the spectrum: it is Black's median voter theorem in the Downs-Hotelling framework.

The basic problem with a majority system when there are three or more candidates is a coordination problem: if the overall preferred candidate is perceived as having negligible chances of being in contention for victory, then voters are likely to vote for their favorite candidate among the two serious candidates. This is what often happens in the United Kingdom for example, where voters avoid "wasting" their vote on the Liberal Democrats as everybody expects either the Conservative or Labour parties to win.

This means that *beliefs* play a key role in elections. This idea that beliefs about the probabilities of close races between the pairs of candidates influences the behavior of the voters as well as the positioning of the candidates was modeled by Myerson and Weber (1993). They, too, proved that when there are more than two candidates in a plurality system, any policy may win in equilibrium. The idea of the proof is that in such a strategic voting set-up, a candidate can be deterred from deviating closer to the median position, because his credibility can be lower in such a configuration and exclude him from the race for victory.

The literature regarding runoff systems is quite limited. Fishburn and Brams (1981) show that a runoff election is not always able to elect the strict Condorcet candidate (corresponding to the candidate positioned the closest to the median). But their analysis holds the positioning of the candidates fixed.[2]

Myerson and Weber (1993) only compare plurality and approval voting systems. The present work extends their model. It proposes a geometric interpretation of their belief

[2] Piketty (1995) develops a model of two successive elections rather than a runoff system.

concept that can be applied to a broad range of commonly observed electoral systems, and therefore enables me to compare them.

Informally stated, the idea of the model is as follows. Each electoral system specifies how the platforms and scores of the candidates give rise to an outcome. A pivot-probability is associated to a pair of outcomes if geometrical analysis shows that they can compete against each other for victory. Beliefs about the pivot-probabilities are given exogenously. I model how the voters' behavior depends on both their preferences and their beliefs. Taking the voters' behavior into account, the parties position in order to maximize their chances to be in government. In each electoral system, equilibria are computed under all possible states of beliefs.

I then illustrate the model by applying it to runoff majority systems. I show that as single-ballot majority systems, runoff systems with three or more parties allow for the implementation of extreme policies. Furthermore, in equilibrium, two candidates are necessarily positioned symmetrically with respect to the median.

The rest of the paper is organized as follows. Section 1 presents the setting of the election. Section 2 analyses the voters' behavior. Section 3 analyses the candidates' behavior and presents the generalized concept of positional equilibrium. Section 4 considers a majority system and obtains a result corresponding to Myerson and Weber's multiple equilibria result in a plurality election. Section 5 considers a runoff majority election where candidates are committed to their positions and illustrates voting behavior with the 2002 presidential French elections. Section 6 presents the main result. Section 7 informally extends the analysis to four parties and shows how the results are modified. Section 8 concludes.

1. The Set-Up

The political space is represented by a one-dimensional [0, 100] segment. There are two sets of agents: voters and candidates.

There are three candidates (denoted 1, 2 and 3). Denote by C the set of candidates and PC the set of subsets of candidates.

The candidates can choose a position anywhere in the political space. They are *electoralist* in the sense that they choose their positions in order to maximize their chances of being in power alone or with another party. The position chosen by candidate i is denoted x_i. Call X the set $[0,100]^3 \subset R^3$. Denote by x positional vector (x_1, x_2, x_3), $x \in X$.

The second set of agents is the set of voters. The voters' ideal points are uniformly distributed over the set $\{0,1, \ldots, 100\}$, with 1/101 of the voters having each ideal in this set. This position t of a voter's bliss point is referred to as the voter's *type*.

The voters do not care which parties are in power, but only care about the implemented policy. The utility of a t-type voter if the implemented policy is z is:

$$U_t(z) = - (z - t)^2$$

If policy z is proposed by some candidate i, this utility is sometimes denoted by $u_i(t)$.

Note that given the voters' preferences distribution and single-peakedness, the median voter's bliss point is well-defined and equals 50.

The present paper focuses on electoral systems where each voter casts a ballot for exactly one candidate, that is, no abstention nor multiple votes are allowed[3]. This will enable me to propose a geometric analysis. Indeed, if s_i denotes the percentage score of candidate i, I have

[3] Myerson and Weber (1993) study elections under plurality, approval and Borda count.

$$s_1 + s_2 + s_3 = 1.$$

The set of all possible results is a two-dimensional simplex, SS, that I represent in a normed plane whose axes are s_1 and s_2. Call s the vector (s_1,s_2,s_3). The set of points corresponding to a constant score S of candidate i, $\{s \in SS : s_i = S\}$ for some constant S with $0 \leq S \leq 1$, is a line, called an iso-s_i line.

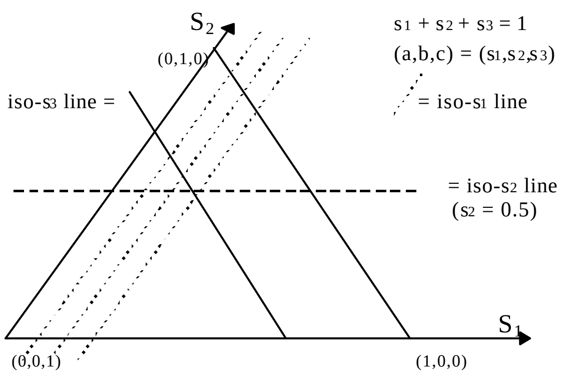

Figure 1: The Score Simplex SS in a voting system with three candidates[4]

A *simple outcome of an election*, (W, z), is an element of PC x [0,100]: its first component is the subset W of candidates who are in power, and its second component is the implemented policy z. Call O the set of elements of PC x [0,100] and PO the set of subsets of O.

An *outcome* is a set of equally likely simple outcomes (one of them being selected later by a coin toss), i.e. an element of PO. Note that all simple outcome singletons are outcomes, selected with probability 1.

An *electoral system* ES with three candidates where voters cast a ballot for exactly one of them is a mapping of 6 variables which maps any point $(s,x) \in$ SS x X on a set of equally likely outcomes: ES: SS x X $\rightarrow$ PO: $(s,x) \rightarrow ES(s,x) \subset PO$.

An electoral system is a mapping rather than a function because some (s,x) can lead to two or more tied outcomes (ties are broken by a fair coin).

Let me call OS the electoral system mapping when x is given: $\forall x$ OS(s) = ES(s,x).

The reciprocal image of an outcome k∈OS(s) by OS is called an *outcome zone*. Thus the outcome zone corresponding to an outcome k, for a given positioning of the candidates, is the set of scores leading to this outcome k.

The *outcome simplex* of an electoral system for a given positioning of the candidates is the score simplex divided in its different outcome zones.

Border assumptions:

1°) for any x ∈ X, all outcome zones are convex sets of the score simplex.

2°) the borders between two outcome zones are straight lines. They have one of the following six directions: the three directions of the lines where the score of one party is constant (i.e. iso-s_i lines, i = 1,2,3) and their perpendiculars

Border assumptions restrict the analysis to electoral systems such that a change in the set of parties in power happens when the score of some set of parties either reaches a threshold or reaches the score of another set of parties.

Call PO(x) the set of possible outcomes (PO) under a given electoral system, given the candidate positioning vector x. Call PPO(x) the set of pairs of elements of PO(x).

There is a *close race* between outcomes k and l where k,l ∈ PO(x), k ≠ l if the distance between point s in the outcome simplex and the frontier between $OS^{-1}(k)$ and $OS^{-1}(l)$ corresponds to at most one vote.

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[4] In order to emphasize the symmetry between the variables, the axes are positioned in a non-orthogonal way. The coordinates of a point are read as suggested.

As there is uncertainty on the realization of the type of each voter, there is some uncertainty about the scores that the candidates will obtain, even if the behavior of any type of voter is known. Call S_i the expected percentage score of candidate i: $S_i = E(s_i)$.

Following Myerson and Weber's terminology, I denote the probability of a close race between two outcomes k and l by p_{kl} and call it a *pivot-probability*.[5]

A *set of beliefs* given an electoral system ES and its corresponding PPO is any function p mapping any positioning x of the candidates on a vector of pivot- probabilities between any two outcomes:

$p: X \to [0,1]^{\# PPO(x)}: x \to p(x)$ with $\Sigma_{(k,l)\in PPO(x)} p_{kl}(x) \leq 1$.

This means that (as in Myerson and Weber) the perception of the relative probabilities of close races between all pairs of outcomes depends on the relative positioning of the candidates. If a candidate i moves from x_i to x_i', the pivot probabilities p(x) are transformed into $p(x_{-i},x_i')$ (where (x_{-i},x_i') denotes the positioning vector if i deviates from x_i to x_i') in a way described by the state of beliefs.

All voters are given such a set of beliefs exogeneously. These beliefs are not necessarily common beliefs.[6]

Connexity assumption: if two outcome zones $OS^{-1}(k)$ and $OS^{-1}(l)$ are not in contact for some positioning x or if they are connected by a single point, then $p_{kl}(x) = 0$.

[5] I implicitly assume (as do Myerson and Weber) that the probability of three outcomes being in close race is infinitesimal in comparison to the probability of a two-outcome tie.

[6] Myerson and Weber assume common beliefs. They also argue that beliefs should be in accordance with the outcome they imply and introduce the so-called *ordering condition*. Our main results are true with or without common beliefs and an adapted ordering condition.

The connexity assumption simply asserts that there are enough voters so that one single voter is not able to make the score vector "jump" from one outcome zone to another if there is a third outcome zone between them.

Call CR(x) the set of pairs of outcomes such that there can be a close race (CR) between them when the candidates hold position x.

By connexity and border assumptions, $p_{kl}(x)$ can be strictly positive only if the frontier between $OS^{-1}(k)$ and $OS^{-1}(l)$ is a non degenerated segment:

$CR(x) = \{(k,l) \in PPO(x)$: the frontier between $OS^{-1}(k)$ and $OS^{-1}(l)$ is a segment$\}$.

As do Myerson and Weber, I assume that all possible pivot-probabilities are strictly positive: $p_{kl}(x) > 0 \ \forall \ (k,l) \in CR(x)$.

2. The Voters' Behavior

When a voter has to decide how to vote, the positions of the candidates (the x vector) are known.

Her behavior will then depend on two things: 1°) her preferences, and how far the candidates are from her bliss point, as she likes to encourage candidates as close as possible to her ideology, and 2°) her beliefs, as she does not want to spoil her ballot.

The balance between these two elements depends on each voter's personal situation and beliefs.

Modeling this balancing is the object of this section and relates to the concepts of expected gain and prospective rating (following Myerson and Weber's terminology).

Let the candidate positioning vector be x and $(k,l) \in CR(x)$. The *expected gain of a type t voter with beliefs p(x) for voting for candidate i regarding the close race between outcomes k and l*, denoted $EG_{i(k,l)}(t)$, is constructed as follows:

1°) If the frontier between $OS^{-1}(k)$ and $OS^{-1}(l)$ is an iso-s_i line for some candidate i:

then one outcome, say k, corresponds to higher scores of candidate i, and the other to lower scores of i, i.e. higher scores of any of the other candidates. Thus,

$EG_{i(k,l)}(t) = p_{kl}(u_k(t) - u_l(t))$, $EG_{j(k,l)}(t) = (1/2)\, p_{kl}(u_l - u_k)$ for $j \neq i$.[7]

2°) If the frontier between $OS^{-1}(k)$ and $OS^{-1}(l)$ is perpendicular to an iso-S_i:

voting for i is not decisive from the point of view of the race between k and l: $EG_{i(k,l)}(t) = 0$. Outcome k corresponds to a higher score for one of the other parties, say j, and lower scores of the last party j', meaning that $EG_{j(k,l)}(t) = p_{kl}\,(u_k - u_l)$; $EG_{j'(k,l)}(t) = p_{kl}\,(u_l - u_k)$.

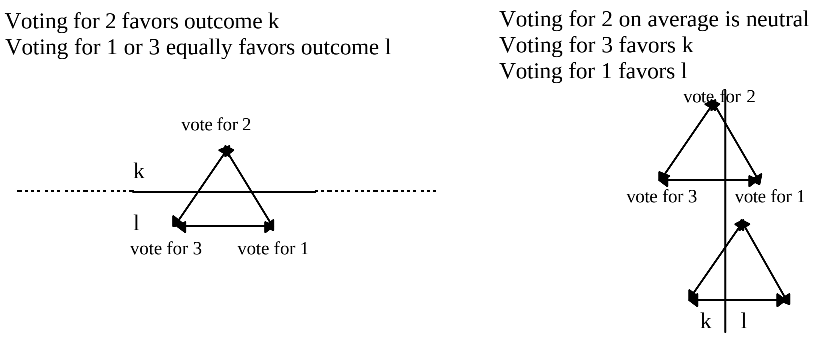

Figure 2: Expected gains relative to a close race between outcomes k and l according to the direction of the border between outcome zones OS^{-1} (k) and OS^{-1}(l)

The overall expected gain for a type t voter of voting for i, called *prospective rating of candidate i* and denoted $PR_i(t)$, is the sum of the expected gains relative to all possible close races: $PR_i(t) = \Sigma_{(k,l)\in CR} EG_{i(k,l)}(t)$.

Behavior of voters:

[7] The expected gain $p_{kl}(u_l - u_k)$ is obtained by voting for any of the two candidates other than i and is therefore split between these two candidates: the "1/2" coefficient guarantees that $\Sigma_{c\in C} EG_c(t) = 0\ \forall t$.

A. If her beliefs are such that all pivot-probabilities are 0, the voter votes sincerely. Ties (in terms of utilities) are broken by a fair coin.

B. If her beliefs are such that at least one pivot-probability is strictly positive, she votes for the candidate who maximizes her prospective ratings. Ties (in terms of prospective ratings) are broken by a fair coin.

Thus, sincere voting is seen as the limit case when a voter believes her vote to change the outcome with probability 0 (case A). Expected utility maximization voting happens when the voter believes this probability to be positive, with the confrontation of her preferences and beliefs expressed in the candidates' prospective ratings.[8] For those voters in B, because the prospective rating is homogenous of degree 1 in p(x), the pivot probabilities for a given positioning of the candidates can be normalized so as to sum to 1 (without loss of generality): $\forall x \in X$, $\Sigma_{(k,l) \in CR(x)} p_{kl}(x) = 1$.

Let me insist on the fact that no voter is playing a game, as the prospective ratings do not depend on the behavior of the other voters (and does not even require them to know the voters' distribution). Voters do not interact directly with each other.[9]

Once candidate positioning x and pivot probabilities p(x) are given, the prospective ratings for all voters are determined. The strategic behavior of the voters determines the expected scores of the candidates, the expected possible outcomes and therefore the expected

[8] The results are robust to suppressing case A, i.e. with at least one pivot probability positive.

[9] Further generalization could allow the voters to play a Nash game.

probability of being in power (alone or with another candidate). These probabilities of winning are endogenous and should not be confused with the exogenous pivot probabilities.

3. The strategies of the candidates - positional equilibrium concepts

The strategies and actions of the candidates, decided before the election takes place, are examined here.

An electoral system has been selected and is common knowledge. A state of beliefs is exogenously given. Each candidate wants to maximize his utility.

I assume that being in power is a cake of size one, shared equally by the winning candidates. Therefore, if P^i_j represents the probability of candidate i winning together with j - 1 other candidates, j = 1, 2, 3 and U_i the utility of party i, I have:

$$U_i = P^i_1 + P^i_2/2 + P^i_3/3 \text{ (and } \Sigma_i U_i = 1).$$

A candidate's only strategic choice is the position he chooses. His choice depends on the electoral system, on the positions of the other candidates and on the state of beliefs.

Now, when optimally choosing his position, he takes the positions of the other candidates as given and computes the probability of being in power in each possible situation, which depends on the pivot probabilities for each of his positions. This means that, in general, the pivot probabilities "out of equilibrium", i.e. the p(y)'s for positions y different from the equilibrium x vector, are not negligible.

I now come to a general definition of positional equilibrium for an election involving three candidates in an electoral system ES.

A situation is a *positional equilibrium* if there exists a set of beliefs function p(y) for each voter such that

1°) each voters maximize her expected utility;

2°) the positioning of the candidates x is a Nash equilibrium given the voters' behaviors under all positioning vectors.[10]

Note that our definition of positional equilibrium is very weak in the sense that I assume no restriction on the state of beliefs function. Beliefs can change with a change of candidates positioning. Myerson and Weber (1993) do not impose either any restriction that would link beliefs across positions: a candidate, considered as a very serious contender under some positioning, might become a sure loser if she, or another candidate, rather picks a slightly different platform. I too choose to be as permissive as possible, since my concept is sufficient to really discriminate across voting systems (as illustrated in a companion paper).

4. Application to a Majority System

This section applies the methodology of the second section to study a relative majority system and shows a very well established result, equivalent to Myerson and Weber (1993)'s theorem 3 for a plurality election in this generalized set-up.

A *relative majority system* is defined as an electoral system in which the party which obtains the highest score, whatever its score, even if it is lower than half of the total of the votes, forms a government alone and implements the policy corresponding to its position x_i.

Thus the *relative majority system* maps (x,s) on outcome $(\{i\}, x_i)$ if $s_i \geq s_j$, $i,j \in C$, $j \neq i$.

[10] Fey (1997) argues that some voting equilibria proposed by Myerson and Weber (1993) are not stable. The present paper does not suffer from this critique, which concerns non Duvergerian equilibria with exogenous candidate positioning.

For any positioning of the candidates, the score simplex is divided in three outcome zones, according to which party has the largest score.

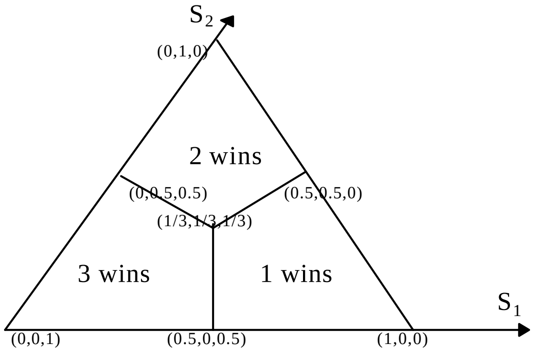

Figure 3: The Outcome Simplex in a relative majority system with three parties

Denote by i the ({i}, x_i) outcome, i = 1,2,3.

The border between any two outcome zones $OS^{-1}(i)$ and $OS^{-1}(j)$ is the segment $\{s: S_i = S_j \geq S_k\}$. Thus $CR(x) = \{(1,2), (1,3), (2,3)\}\ \forall x \in X$ and the possibly positive pivot-probabilities are p_{12}, p_{13} and p_{23}.

The expected gains are easy to construct: as all the outcome zones borders are perpendicular to an iso-score line, each p_{ij} affects only candidates i and j: $EG_{i(i,j)}(t) = p_{ij}(u_i - u_j)$, $\forall i,j,\ i \neq j$, $EG_{i(k,l)}(t) = 0$, $\forall i,k,l,\ i \neq k,\ i \neq l,\ k \neq l$. And I obtain the following prospective ratings: $PR_i(t) = p_{ij}[u_i(t) - u_j(t)] + p_{ik}[u_i(t) - u_k(t)]$, $\forall i,k,l,\ i \neq k,\ i \neq l,\ k \neq l$.

This electoral system corresponds to Myerson and Weber (1993) plurality election, and my notations are chosen so as to coincide with theirs.

In this set-up, I have the equivalent to Myerson and Weber' theorem 3:

Proposition 1: In a one round relative majority system, for any z strictly between 0 and 100, policy z can be implemented in positional equilibrium.

Proposition 1 also holds under common beliefs.

Proof of proposition 1: see the appendix. The idea of the proof is similar to Myerson and Weber's theorem 3. The authors show their result by constructing an example of an equilibrium whose winning policy is any z in the political spectrum: 1 positions at z, 2 at 50 and 3 at 100 - z/2. They consider a situation with the common belief that $p_{13} = 1$ while p_{12} and p_{23} are negligible: everybody believes that 2 has no chance of winning the election, avoids wasting her vote for this "more moderate" candidate and chooses between 1 and 3, giving the victory to the more centrist of them: candidate 1.

This situation is an equilibrium for the following reason. Candidate 3 considers adopting a more centrist position to win the election. But he can be dissuaded from doing so, if beliefs in the alternative configuration (where he is closer to the center) give him no credibility as a serious contender: if he picks a central platform, the beliefs are $p_{12} = 1$ rather than $p_{13} = 1$.

Myerson and Weber's multiple equilibria result in a plurality election is easily interpreted in relative majority election where there are three parties: the largest party, whatever its score, even if it is lower than half of the total of the votes, gets the power.

I show in the appendix that proposition 1 still holds if I add a coalition-proofness refinement à la Bernheim, Peleg and Whinston in the positioning game.

As already mentioned, UK (at least before New Labour) is a good illustration of this result.

5. Application to Runoff Systems

A frequently observed electoral system is the double-ballot system, or majority runoff system. There are several versions of a majority runoff system. This paper adopts the following simple stylized definition.

A *runoff majority system* is defined as an electoral system in which, at the first ballot, voters cast a ballot for one of the candidates. If one candidate commands an absolute majority of votes, he wins the election. If no candidate receives more than half the votes, there is a runoff election between the first two candidates. The winner of the head-to-head runoff wins the election.

Modified definition of simple outcome: a simple outcome is a simple outcome as defined in section 1, indexed by its history: if k is an outcome as defined before, let k denote "outcome k taking place after the first round" and k_i "outcome k taking place after a second round against outcome i".

I assume the voter's utility is independent of whether the winning outcome is selected at the first or at the second round: $u_k(t) = u_{ki}(t)$.

Now there are possibly two ballots, the following question is raised: are the candidates allowed to change their positions between the first and the second round? The answer to this question influences the strategic behavior of both the voters and the parties. Here I consider *committed candidates*: they do not modify their platforms after the first round.[11]

Now the election is possibly divided in two rounds, the analysis of the mapping and the pivot-probabilities seems more complicated. Nevertheless, the analysis of the second round (if there is any) is very simple: as the candidates do not relocate, the positioning vector x is given. The voting behavior of the voters in the second round is trivial, as there are two candidates: they vote for the candidate the closer to their ideal point, who provides them with the higher expected gain.

[11] The case of a runoff system with uncommitted parties could also be developed.

This means that if a second round opposes any two candidates i and j, the candidate closer to the median wins. This is expressed as follows, where d(x,y) denotes the Euclidian distance between x and y on the [0,100] segment:

$\forall i \neq j$, $i,j \in C$, let *ivsj(x)* denote outcome i_j if $d(x_i,50) < d(x_j,50)$; outcome j_i if $d(x_j,50) < d(x_i,50)$ and outcome $\{i_j, j_i\}$ if $d(x_i,50) = d(x_j,50)$.

The *runoff majority system mapping* is the following:

$\forall i,j,j' \in C$, $i \neq j$, $i \neq j'$, $j \neq j'$:

$ES(x,s) \ni i$ if $S_i \geq 50\%$;

$ES(x,s) \ni ivsj$ if $S_k \leq 50\%$ $\forall k$, $S_i > S_{j'}$ and $S_j > S_{j'}$;

$ES(x,s) \ni \{ivsj, ivsj'\}$ if $S_k \leq 50\%$ $\forall k$, $S_i > S_{j'} = S_j$;

$ES(x,s) \ni \{ivsj, ivsj', jvsj'\}$ if $S_k \leq 50\%$ $\forall k$, $S_i = S_{j'} = S_j$.

There are six outcome zones: three where one party i has more than 50%, leading to i, and three where none of them has 50% and a runoff must take place:

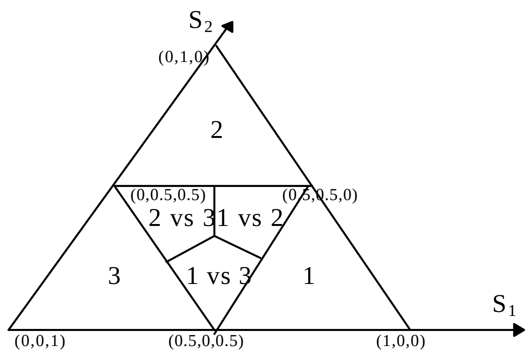

Figure 4: the outcome simplex under majority runoff system

The behavior of the voters in the first round decides the outcome: once the first round results are given, the voters know what will happen in the second round.

CR(x) = {(1,1vs2(x)), (1,1vs3(x)), (2,1vs2(x)), (2,2vs3(x)), (3,1vs3(x)), (3,2vs3(x))} and the pivot-probabilities are of the type $p_{i,ivsj}$, and $p_{ivsj,ivsk}$, i,j,k $\in$ C, i $\neq$ j, i $\neq$k, j $\neq$ k. Expected gains are constructed: pivot-probabilities of the type $p_{i,ivsj}$ refer to iso-S_i borders: $EG_{i(i,ivsj)} = p_{i,ivsj}(u_i - u_{ivsj})$ and $EG_{k(i,ivsj)} = EG_{j(i,ivsj)} = (1/2)p_{i,ivsj}(u_{ivsj} - u_i)$ while $p_{ivsj,ivsk}$ type pivot-probabilities refer to the perpendicular direction: $EG_{i(ivsj,ivsk)} = 0$, $EG_{k(ivsk,ivsj)} = p_{ivsk,ivsj}(u_{ivsk} - u_{ivsj})$ and $EG_{j(ivsk,ivsj)} = p_{ivsk,ivsj}(u_{ivsj} - u_{ivsk})$. $PR_i(t) = \Sigma_{h\in CR}EG_{ih}(t)$, i = 1,2,3.

Here as in a plurality election, voters can encourage their second best to win against their worst choice. The mechanics are more complex though. Consider for example a voter who prefers candidate 1 to 2 and 2 to 3, with 2 more moderate than 1 and 1 more moderate than 3. Then that voter can encourage 3 to reach the second round so as to have him lose against 1 (while 2 would win against 1).

Let me illustrate the voting behavior in a run-off election with the 2002 presidential election in France. How could LePen possibly obtain so many votes in the first round and so few in the second round? Consider left-wing Jospin and right-wing Chirac as the most serious contenders, together with (minor candidates and) far-right LePen as a focal underdog hated by a wide majority of the population.

Assume Jospin (J), Chirac (C) and LePen (LP) positioned at 40, 60 and 90: x = (40, 60, 90).

The outcome simplex is

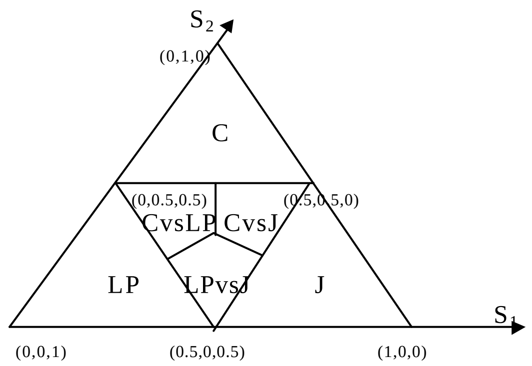

Figure 5: the outcome simplex – France 2002

Given their positioning, a second round between LePen and Chirac or Jospin would lead to Chirac or Jospin's victory with certainty, while the outcome of a second round between Chirac and Jospin is uncertain.

Assume you are a Chirac fan. Is it possible you vote for LePen in the first round? Yes, if you believe Chirac will reach the second round anyway (i.e. $p_{CvsJ,CvsLP}$ is the only positive pivot-probability - normalized to 1). Since he is sure to win against LePen, but much less so against Jospin, LePen is the candidate you should push in the first round.

Similarily, it is possible for a Jospin fan to vote for LePen: If you believe $p_{CvsJ,JvsLP}$ is the only positive pivot-probability, you should push LePen in order to have Jospin Vs. LePen in the second round.

The actual results of the 2002 French presidential elections can be explained in the framework of this paper in, say, the following scenario:

The electorate was partitioned into three groups (uniformly distributed among the ideological space for simplicity) regarding their perceptions of the election: Group 1 (50% of the population) think they can influence the final outcome with probability 0. Group 2 (25%)

perceive that $p_{CvsJ,CvsLP}$ is the only positive pivot-probability. Group 3 (the remaining 25%) perceive that $p_{CvsJ,JvsLP}$ is the only positive pivot-probability.

In group 1 all people vote sincerely. In group 2 J and LP supporters vote for their preferred candidate while C supporters vote for C. In group 3, C and LP supporters vote for C while J supporters vote for LP. Aggregate votes in the first round lead to 50% > $S_C > S_{LP} > S_J$. In the second round $S_C \approx 85\% > S_{LP} \approx 15\%$. Thus a candidate like LePen can reach the second round and then score less than 20%.

I am not suggesting that my model explains *all* votes for LePen, but I do believe that it is useful to note that the results can also be explained in a framework where the goal of voting is to influence the outcome rather than to "communicate" or to "protest". The classic explanation, in this vein, is that voters used the first round as an opportunity to protest against corruption around Chirac and mainstream politics.

6. The Result

Once the voters' behavior is understood, the strategic positioning of the candidates can be examined. As in a plurality election, a strictly most extremist candidate cannot win, since he is the worst choice for an absolute majority in both rounds. The dynamics are different though, as to be selected for a second round is only useful for the more moderate of the two candidates. Nevertheless, the following result shows that here as in a plurality election, adopting a moderate platform does not guarantee winning, because of the manipulative power of beliefs.

Proposition 2: In a majority runoff system equilibrium, any policy can be implemented in equilibrium and at least two candidates are positioned symmetrically with respect to the median.

Proposition 2 also holds under common beliefs (and the following proof is identical).

Proof of proposition 2:

Lemma 1: Under any state of beliefs, if $x_1 \leq x_2 < x_3 \leq 50$, then 3 wins with probability 1

Proof of lemma 1:

In such a situation the prospective ratings are as follows:

$PR_1(t) = p_{1,1vs2}(u_1 - u_2) + p_{1,1vs3}(u_1 - u_3) + p_{1vs2,2vs3}(u_2 - u_3) + (1/2)p_{2,2vs3}(u_3 - u_2);$

$PR_2(t) = (1/2)p_{1,1vs2}(u_2 - u_1) + (1/2)p_{1,1vs3}(u_3 - u_1) + p_{1vs2,1vs3}(u_2 - u_3) + p_{2,2vs3}(u_2 - u_3);$

$PR_3(t) = (1/2)p_{1,1vs2}(u_2 - u_1) + (1/2)p_{1,1vs3}(u_3 - u_1) + p_{1vs2,1vs3}(u_3 - u_2) + (1/2)p_{2,2vs3}(u_3 - u_2) + p_{1vs2,2vs3}(u_3 - u_2).$

Then $PR_3(t) > PR_1(t)$ and $PR_3(t) > PR_2(t)$ for any t such that $t > (x_2 + x_3)/2$.

Thus 3 wins with probability 1.

Lemma 2: Assume $x_1 < x_2 = x_3$, with $d(x_1,50) > d(x_2,50)$. Then 3 and 2 win with probability ½.

Proof of lemma 2:

$PR_1(t) = p_{1,1vs2}(u_1 - u_2) + p_{1,1vs3}(u_1 - u_2);$

$PR_2(t) = (1/2)p_{1,1vs2}(u_2 - u_1) + (1/2)p_{1,1vs3}(u_2 - u_1);$

$PR_3(t) = (1/2)p_{1,1vs2}(u_2 - u_1) + (1/2)p_{1,1vs3}(u_2 - u_1).$

Then $PR_3(t) = PR_2(t) > PR_1(t)$ for any t such that $t > (x_1 + x_2)/2$, leading to a secound round between 1 and 2 or 3 with equal probabilities, with 1 losing if more extremist, and winning with probability ½ otherwise.

<u>Lemma 3: If there is a strictly most extremist candidate, he wins with probability 0.</u>

Proof of lemma 3:

Assume there is a strictly most extremist candidate, call him 1 and assume by symmetry that $x_1 < 50$ and $x_1 < x_2 \leq x_3$, with $x_3 < 100 - x_1$.

Assume that 1 win with a strictly positive probability.

Then by lemma 2, $x_1 < x_2 < x_3$ and by lemma 1, $50 < x_3$.

Then one candidate among {2,3} wins with a probability strictly smaller than 1/2. But then this candidate deviates to the other one, and will win with probability 1/2 by lemma 2. Thus in equilibrium 1 must win with probability 0.

<u>Lemma 4: A situation with $x_1 < x_2 = x_3$, with $d(x_1,50) < d(x_2,50)$ is not an equilibrium.</u>

Proof of lemma 4:

Assume it is an equilibrium. Then 1 wins with probability 1, otherwise he would deviate to 50 and win with probability 1 by lemma 1.

Therefore 2 and 3 are sure losers. But then 2 would deviate to 1 and win with probability ½ by lemma 2.

Proof of proposition 2:

1) A situation with all parties positioned at the median is an equilibrium: all prospective ratings are 0 and all parties win with probability 1/3, while if one party deviates it wins with probability 0 by lemma 2.

2) A situation where two parties are positioned symmetrically around 50 with the third party at 50 is a positional equilibrium.

Consider that $d(x_1,50) = d(x_2,50)$ with $x_3 = 50$, $x_1 \leq x_2$ and any state of beliefs such that $p_{1,1vs2}(x_1,x_2,z) = p_{2,1vs2}(x_1,x_2,z) \cong 1/2$ for any z, $p_{1vs3,1vs2}(z, x_2, 50) \cong 1$ if $z < 50$ and $z \neq x_1$, and $p_{2vs3,1vs2}(x_1, z, 50) \cong 1$ if $z > 50$ and $z \neq x_2$.

$PR_1 = 1/2.(p_{1,1vs2} + 1/2.p_{2,1vs2})(u_1 - u_2)$ + negligible terms > 0 for any t < 50;

$PR_1 = 1/2.(p_{1,1vs2} + 1/2.p_{2,1vs2})(u_1 - u_2)$ + negligible terms < 0 for any t > 50;

$PR_2 = 1/2.(p_{2,1vs2} + 1/2.p_{1,1vs2})(u_2 - u_1)$ + negligible terms > 0 for any t > 50;

$PR_2 = 1/2.(p_{2,1vs2} + 1/2.p_{1,1vs2})(u_2 - u_1)$ + negligible terms < 0 for any t < 50;

$PR_3 = 1/4.(p_{1,1vs2} - p_{2,1vs2})(u_2 - u_1)$ + negligible terms $\cong 0$ for any t;

Thus, $50\% > S_1 = S_2 > S_3$.

This situation is a positional equilibrium, for all parties can be prevented from deviating.

Indeed, candidate 1 is dissuaded as follows (a similar argument applies to 2): if 1 deviates to z > 50 then his probability to win is 0 as 3 then wins with probability 1 by lemma 1. If 1 deviates to 50, then he wins with probability 1/2 by lemma 2. If he deviates to z < 50, the belief that $p_{1vs3,1vs2} \cong 1$ imply the prospective ratings become either $PR_1 \cong 0$, $PR_2 = u_2 - u_3$ + negligible terms $\cong -PR_3$ if 2 is more moderate than 1, or $PR_1 \cong 0$, $PR_2 = u_1 - u_3$ + negligible terms $\cong -PR_3$ otherwise and 3 wins in the first round.

Candidate 3 can be deterred from moving by simply $p_{1,1vs2}(x_1,x_2,z) = p_{2,1vs2}(x_1,x_2,z) \cong 1/2$ for any z, as the prospective ratings are maintained.

Note that it is not necessary that some party occupy the median position.

3) It is now sufficient to note that if no pair of candidates is positioned symmetrically with respect to the median, the most extremist candidate is a sure loser and wins with a strictly positive probability by joining the most moderate party by lemma 2.

The appendix shows that the result is robust to the introduction of coalition-proofness.

Therefore, neither single nor double ballots majority systems guarantee political moderation. The equilibria look quite different though: while about any configuration of candidates is possible under single ballot elections, an interesting symmetry occurs with two ballots.

France is a natural illustration to proposition 2. I observe no strong convergence, the two serious contenders being usually one moderate left-wing candidate and one moderate right-wing candidate, with some power alternance over time and one smaller party (typically either right-wing or far-right).

Note that if repositioning by the candidates is allowed between the second and the third rounds, a median voter result can easily be proven.

7. Extension to Four Candidates

The purpose of this section is to show informally how the most important results change with the number of parties.

If there are four parties, the score simplex is three-dimensional. Outcome zones are three dimension convex volumes. Outcomes are possibly in close race if they are connected by a plane surface. Pivot probabilities, prospective ratings and equilibrium concepts are constructed in a way similar to the 3 party case (see the appendix).

Consider first a majority system. The electoral mapping is constructed exactly as in the three party case. Pivot-probabilities are still of the form p_{ij}, $\forall$ i ≠ j. And p_{ij} only affects PR_i and PR_j. Unsurprisingly, Proposition 1 still holds if there are 4 parties:

Proposition 1 bis: In a majority system with four candidates, any policy can be implemented in equilibrium.

Proof: Similar to proposition 1 and Myerson and Weber's corresponding result. Available from the author. This result easily generalizes to any n ≥ 3.

In a runoff system, the three candidate case can be considered as very particular: since the first round selects two parties, it equivalently gets rid of one party. This is no longer true with four candidates.

The electoral mapping is constructed exactly as in the three party case. Pivot-probabilities are still of the form $p_{i,i+j}$ and $p_{i+j,i+k}$ $\forall i, \forall$ j ≠ i, k ≠ i, k ≠ j. Pivot probability $p_{i,i+j}$ affects the prospective ratings relative to all parties while $p_{i+j,i+k}$ only affects PR_j and PR_k.

Proposition 2 bis: In a runoff system with 4 committed candidates, any policy can win in equilibrium.

Proof: Similar to proposition 2. Available from the author. This result extends to any number of parties and is robust to coalition-proofness.

The main implication of propositions 2 and 2bis is that runoff systems are unable to guarantee more moderation than a majority system if candidates are not able to reposition between rounds.

With regards to my results, the hope of Piketty (1995) to allow for an efficient separation of the communication and decision making purposes of voting by implementing a runoff election seems to be vain: the first round can be decisive and therefore voting in the first round is highly strategic (as shown in the example pages 17-18).

7. Conclusion and Discussion

This paper constructs a general model allowing the study of a large range of electoral systems with three and four parties. This spatial model, with endogenous positioning of the candidates and sophisticated voters, enables to compare equilibria in a unified set-up potentially involving a wide range of electoral systems. My basic model develops a three party set-up. The analysis is extended to a four party set-up in the appendix.

The overall results of the paper can be summarized as follows.

My result regarding a majority system includes Myerson and Weber (1993)'s result of multiple equilibria in a plurality system. It is interpreted as a result of inability for a majority system to guarantee any moderation of policy.

I modeled a runoff system and showed that such systems are not able to guarantee moderate outcomes either. All equilibria present an interesting symmetry.

In a companion paper I examine proportional systems and show how different coalition formation rules can be incorporated into the theoretical analysis. I show that they crucially affect the moderation capacity of a proportional system.

The belief concept I used is also driving the positional equilibrium concept in Myerson and Weber (1993). It is central to the results and should be briefly discussed. It is weak in the following sense: the beliefs are allowed to vary without restriction with the change of positioning of a party. This is sometimes unrealistic and implies that beliefs are very powerful. Nevertheless, this concept of beliefs is sufficient to discriminate across systems, as my results attest. What makes the results change across systems is the structure of possible beliefs themselves.

In any case, I think that the concept of pivot-probability remains very natural and relevant in this kind of set-up. Future research should try to explore further this belief concept and to find an appropriate way of imposing more structure on the state of belief function, in order to avoid caricatured beliefs.

This model is a first attempt to compare the major electoral systems in a unified set-up and brings realistic new results. It is quite general and could be used to study different versions of these systems, or different systems, as long as voters are assumed to cast a ballot for exactly one candidate or party.

A weakness of this paper certainly is the exogeneity of the number of parties. I neglect the possibility of a correlation between the number of parties and the electoral system, which should be taken into account when comparing voting systems.

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Appendix 1: Technical Proofs

Proof of proposition 1 (including coalition-proofness):

First consider the positional equilibrium case. For reasons of symmetry, I only construct an example of an equilibrium whose outcome z is anywhere between 1 and 50: x = (z, 50, 100 - z/2). Consider a state of beliefs such that $p_{13}(z,x_2,100 - z/2) \approx 1\ \forall x_2$, $p_{12}(z,x_2,x_3) \approx 1\ \forall x_2$ and $\forall x_3 \neq 100 - z/2$, and any pivot-probabilities otherwise. As $p_{13}(x) \approx 1$, party 1 wins.

Candidate 1 does not deviate, as he is the winner. 2 does not deviate since $p_{13}(z,x_2,100 - z/2) \approx 1\ \forall x_2$. 3 does not deviate since $p_{12}(z,x_2,x_3) \approx 1\ \forall x_2$ and $\forall x_3 \neq 100 - z/2$.

Further more, this situation is coalition-proof if the state of beliefs is also such that:

$$p_{13}(z,x_2,x_3) \approx 1\ \forall\ x_2 \neq 50,\ \forall\ x_3 \neq 100 - z/2 \text{ and } x_3 \neq z \qquad \text{(a1)},$$

$$p_{12}(z,x_2,x_3) \approx 1\ \forall\ x_2 \neq 50,\ x_3 = z \qquad \text{(a2)}.$$

Indeed the only possible deviating coalition is formed by 2 and 3 but 1°) a deviation to x' = (z, x_2 ,x_3) if $x_3 \neq z$ does not make 2 better off; 2°) a deviation to x' = (z,x_2,z) with $x_2 \neq 50$

does not make 3 better off; 3°) a deviation to x' = (z,z,z) is not self-enforcing either: it implies P = (1/3,1/3,1/3) and 2 deviates back to 50 and wins for sure (since all pivot-probabilities are strictly positive).

Proof of coalition-proofness in proposition 2:

This situation is possible in coalition-proof equilibrium, for example if the beliefs also satisfy

$p_{1,1vs3}(x_1,z,100-x_1) = p_{3,1vs3}(x_1,z,100 - x_1) \cong 1/2$ for any z, (a3);

$p_{1vs3,1vs2}(x_1, x_1, z) \cong 1$ for any z with $x_1 < z < 100 - x_1$ (a4);

$p_{1vs3,1vs2}(z, x_2, 50) \cong 1$ if $z < 50$ and $p_{2vs3,1vs2}(x_1, z, 50) \cong 1$ if $z > 50$.

Indeed, a self-enforcing coalition including 1 or 2 requires him to win with a probability strictly superior to 1/2.

Thus, for reasons of symmetry, I will consider any collective deviation by 2 and 3 leading to $P_2 > 1/2$ and $P_3 > 0$, with $x'_2 \neq x_2$, $x'_3 \neq x_3$ and x_1 fixed and show that it is not self-enforcing. By lemma 3, I must have $x_1 \leq x'_2 \leq 100 - x_1$ and $x_1 \leq x'_3 \leq 100 - x_1$.

If $x_1 = x'_2 = x'_3$, $P_i = 1/3$ for any candidate.

If $x_1 = x'_2$ and $x'_3 = 100 - x_1$, $P_2 = 0$ by (a3).

If $x_1 = x'_2 < x'_3 < 100 - x_1$, then $P_3 = 1$ by (a4).

If $x_1 = x'_3 < x'_2 < 100 - x_1$, 3 must win with probability ≥ 1/2, since otherwise he deviates further to x'_3 and wins with probability 1/2 by lemma 2. This contradicts the condition that 2 wins with probability > 1/2.

If $x_1 < x'_2 = x'_3 < 100 - x_1$, then 2 and 3 win with probability 1/2 by lemma 2.

If $x_1 < x'_2 < x'_3 < 100 - x_1$, or $x_1 = x'_3 < x'_2 < 100 - x_1$, the one who wins with probability < 1/2 deviates to the other and wins with probability 1/2 by lemma 2.

I just showed that there is no self-enforcing deviation by 2 and 3, and can show a symmetric result for 1 and 3. There is no collective deviation by 1 and 2 since the situation is Pareto-efficient for {1,2}: $U_1 + U_2 = 1$.

Appendix 2: The case of 4 Candidates

Formalization with four candidates:

Call C the set of candidates {1,2,3,4} and X the positioning vector of the candidates. Consider the same world as in section 1.

If s_i denotes the percentage score of candidate i, I have $s_1 + s_2 + s_3 + s_4 = 1$ and the set of all possible results is a three dimensional simplex, SS. It can be represented by a tetrahedron within a three-dimensional non orthogonal repere. Call s the vector (s_1,s_2,s_3,s_4). The set of points corresponding to a constant score S of candidate i, $\{s \in SS : s_i = S\}$ for some constant S with $0 \leq S \leq 1$, is a plane surface, called an iso-s_i plane.

An *electoral system* ES is a mapping of 8 variables which maps any point $(s,x) \in SS \times X$ on a set of equally likely outcomes: $ES: SS \times X \to GK$: $(x,s) \to ES(x,s) \subset GK$.

Convexity assumption: all outcome zones are convex sets.

Border assumption: the borders between two outcome zones are plane surfaces. They have one of the following directions: the directions of the planes where the score of one party is constant (i.e. iso-s_i lines, i = 1,2,3), the planes perpendicular to two such directions and the plane such that the sum of two scores sum up to a constant number.

Connexity assumption: if for some positioning x two outcome zones $OS^{-1}(k)$ and $OS^{-1}(l)$ are not in contact, if they are connected by a single point, or a single segment, then $p_{kl}(x) = 0$.

Lemma: For any $x \in X$ and any $(k,l) \in PO(x)$, $p_{kl}(x)$ can be strictly positive only if the frontier between $OS^{-1}(k)$ and $OS^{-1}(l)$ is a place surface.

$CR(x) = \{(k,l) \in PPO(x)$: the frontier between $OS^{-1}(k)$ and $OS^{-1}(l)$ is a plane surface$\}$.

The *expected gain for a type t voter of voting for candidate i relative to the close race between outcomes k and l*, denoted $EG_{i(k,l)}(t)$, is constructed as follows:

1°) If the frontier between $OS^{-1}(k)$ and $OS^{-1}(l)$ is an iso-s_i surface plane for some candidate i:

then one outcome, say k, corresponds to higher scores of candidate i, and the other to lower scores of i, i.e. higher scores of any of the other candidates. Thus, $EG_{i(k,l)}(t) = p_{kl}(u_k(t) - u_l(t))$, $EG_{j(k,l)}(t) = 1/3.\ p_{kl}(u_l - u_k)$ for $j \neq i$.

2°) If the frontier between $OS^{-1}(k)$ and $OS^{-1}(l)$ is perpendicular to an iso-S_i and an iso-S_h:

voting for i or h is not decisive from the point of view of the race between k and l: $EG_{i(k,l)}(t) = EG_{h(k,l)}(t) = 0$. Outcome k corresponds to a higher score for one of the other parties, say j, and lower scores of the last party j', meaning that $EG_{j(k,l)}(t) = p_{kl}\ (u_k - u_l)$; $EG_{j'(k,l)}(t) = p_{kl}\ (u_l - u_k)$.

2°) If the frontier between $OS^{-1}(k)$ and $OS^{-1}(l)$ is a plane where $S_i + S_h$ = a constant:

then one outcome, say k, corresponds to higher scores of candidate i and another candidate, say g, and the other to higher scores of h, and some other candidate, say j. Thus, $EG_{i(k,l)}(t) = EG_{g(k,l)}(t) = 1/2.p_{kl}(u_k(t) - u_l(t))$, $EG_{h(k,l)}(t) = Eg_{i(k,l)}(t) = 1/2.\ p_{kl}(u_l - u_k)$ for $j \neq i$.

Majority System:

The *relative majority system mapping* maps (x,z) on outcome (i, x_i) if $s_i \geq s_j$, i,j $\in$ C, j $\neq$ i. The border between any two outcome zones $OS^{-1}(i)$ and $OS^{-1}(j)$ is the plane surface {s: $S_i = S_j \geq S_k$}. Thus CR(x) = {(1,2), (1,3), (2,3), (1,4),(2,4),(3,4)} $\forall x \in$ X and the possibly positive pivot-probabilities are the p_{ij}'s, $\forall$i,j $\in$C, i $\neq$ j.

As all the outcomes zones borders are perpendicular to an iso-score place, each p_{ij} affects only candidates i and j: $EG_{i(i,j)}(t) = p_{ij}(u_i - u_j)$, $\forall$i,j, i $\neq$ j, $EG_{i(k,l)}(t) = 0$, $\forall$i,k,l, i $\neq$ k, i $\neq$ l, k $\neq$ l. And I obtain the following prospective ratings: $PR_i(t) = \Sigma_j\, p_{ij}[u_i(t) - u_j(t)]$, $\forall$i, $\forall$j $\neq$ i.

Runoff system with committed parties

The analysis of the second round is very simple, as in section 5: if a second round opposes any two candidates i and j, the candidate closer to the median wins.

The *runoff majority system with committed parties mapping* is the following:

$\forall$i,j,j',i' $\in$ C, i $\neq$ j, i $\neq$j', j $\neq$ j', i' $\neq$ i, i' $\neq$j, i' $\neq$ j':

ES(x,s) $\ni$ i if $S_i \geq 50\%$;

ES(x,s) $\ni$ ivsj if $S_k \leq 50\%$ $\forall$k, $S_i > S_j > S_{j'}$ and $S_j > S_{i'}$;

ES(x,s) $\ni$ {ivsj,ivsj'} if $S_k \leq 50\%$ $\forall$k, $S_i > S_{j'} = S_j > S_{i'}$;

ES(x,s) $\ni$ {ivsj,ivsj',ivsi'} if $S_k \leq 50\%$ $\forall$k, $S_i > S_{j'} = S_j = S_{i'}$;

ES(x,s) $\ni$ {ivsj,ivsj',jvsj'} if $S_k \leq 50\%$ $\forall$k, $S_i = S_{j'} = S_j > S_{i'}$;

ES(x,s) $\ni$ {ivsj,ivsj',jvsj',ivsi',jvsi',j'vsi'} if $S_k \leq 50\%$ $\forall$k, $S_i = S_{j'} = S_j = S_{i'}$;

CR(x) = {(1,1vs2(x)), (1,1vs3(x)), (1,1vs4(x), (2,1vs2(x)), (2,2vs3(x)), (2,2vs4(x)), (3,1vs3(x)), (3,2vs3(x)), (3,3vs4(x)), (4,4vs1(x)), (4,4vs2(x)), (4,4vs3(x))}. And therefore the pivot probabilities are of the type $p_{i,ivsj}$, and $p_{ivsj,ivsk}$, i,j,k $\in$ C, i $\neq$ j, i $\neq$k, j $\neq$ k.

Expected gains are constructed: Pivot probabilities of the type $p_{i,ivsj}$ refer to iso-S_i planes: $EG_{i(i,i+j)} = p_{i,i+j}(u_i - u_{ivsj})$ and $EG_{k(i,ivsj)} = EG_{j(i,ivsj)} = EG_{l(i,ivsj)} = (1/3)p_{i,ivsj}(u_{ivsj} - u_i)$ while $p_{ivsj,ivsk}$ type pivot probabilities refer to the perpendicular direction: $EG_{i(ivsj,ivsk)} = EG_{l(ivsj,ivsk)} = 0$, $EG_{k(ivsk,ivsj)} = p_{ivsk,ivsj}(u_{ivsk} - u_{ivsj})$ and $EG_{j(ivsk,ivsj)} = p_{ivsk,ivsj}(u_{ivsj} - u_{ivsk})$. $PR_i(t) = \Sigma_{h\in CR}EG_{ih}(t)$, $i = 1,2,3,4$.