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Are Non-Fundamental Equilibria Learnable in Models of Monetary Policy?[¤]

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Abstract

Recent models of monetary policy can have indeterminacy of equilibria, which is often viewed as a di±culty of these models. We consider the signi¯cance of indeterminacy using the learning approach to expectations formation. We employ expectational stability as a selection criterion for di®erent equilibria and derive the expectational stability and instability conditions for forward-looking multivariate models, both without and with lags. The results are applied to several monetary policies.

Key words: Adaptive learning, stability, sunspots, monetary policy
JEL classi¯cation: E52, E31, D84.

1 Introduction

In recent years there has been a large amount of research studying the performance of alternative monetary policies in dynamic macroeconomic settings; for example, see the survey (Clarida, Gali, and Gertler 1999) and the papers in the 1999 Special Issue of the Journal of Monetary Economics and in the volume (Taylor 1999). A di±culty has emerged in this literature: many recent models of monetary policy are plagued by the problem of indeterminacy, i.e. there are multiple, even continua of rational expectations equilibria (REE).[1]

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[1] These include bubbles or sunspots, see e.g. the discussions in (Kerr and King 1996), (Bernanke and Woodford 1997), (Woodford 1999), (Clarida, Gali, and Gertler 1999), (Bullard and Mitra 2002), and (Carlstrom and Fuerst 2000). See (Evans and Honkapohja 2003a) for a review.

The issue of indeterminacy can be important from a practical point of view. Pursuit of optimal monetary policy on the part of the central bank or, flexible inflation targeting in the sense used by (Svensson 1999), implies that the instrument of monetary policy, the short-term nominal interest rate, should respond to inflation forecasts; see (Clarida, Gali, and Gertler 1999). (Clarida, Gali, and Gertler 1998) provide evidence that monetary policy in a number of industrialized countries (like Germany, Japan, and the U.S.) has been forward-looking since 1979.[2]

A number of theoretical studies have considered the issue of indeterminacy with interest rate rules and different views have been taken on this problem. (Bernanke and Woodford 1997) have argued against inflation forecast targeting on the basis of indeterminacy - these rules may lead to too much volatility in inflation and output which any central bank ought to avoid; see also (Woodford 2003b) and (Woodford 2003a). At the other end of the spectrum, indeterminacy is viewed as an unimportant curiosum. For instance, in (McCallum 2001a), (McCallum 2001b) and (McCallum 2003) the use of the minimal state variable (MSV) solution is advocated for applied analysis. To fix terminology, we will use the terms fundamental vs. non-fundamental equilibria to distinguish between the MSV and other REE.

A recent paper taking indeterminacy as an empirically relevant possibility is (Clarida, Gali, and Gertler 2000). They estimate a forward-looking policy reaction function for the postwar U.S. economy, both before and after the appointment of Paul Volcker as Fed Chairman in 1979. They conclude that monetary policy in the pre-Volcker era was compatible with the possibility of bursts of inflation and output that resulted from self-fulfilling changes in expectations of the private sector. In this way monetary policy of the Federal Reserve contributed to the high and volatile inflation of the 1960s and 1970s. Analytically, the pre-Volcker period is modelled as a non-fundamental REE. In contrast, monetary policy in the Volcker-Greenspan era is compatible with the existence of a unique fundamental equilibrium delivering low and stable inflation.

In this paper we take a new perspective on the problem of indeterminacy induced by monetary policy by introducing a selection criterion among the REE to narrow down the set of plausible equilibria. We use the adaptive learning approach to expectation formation that has recently gained some popularity.[3] In general terms the learning approach suggests that expectations might not always be fully rational, and the REE of interest should satisfy a natural stability criterion in expectations formation. If economic agents make forecast errors and adjust their forecast functions over time, the economy will reach an adaptively stable or learnable REE asymptotically where these forecast errors eventually disappear. In contrast, adaptively unstable REE will not emerge as an outcome from such adjustment processes.

Even though our motivation is primarily an analysis of recent models of monetary

[2]Recent evidence by (Alesina, Blanchard, Gali, Giavazzi, and Uhlig 2001) also suggests that the European Central Bank (ECB) may have been forward-looking. Moreover, a number of inflation-targeting central banks like those in England, Canada, and New Zealand are forward-looking in practice.

[3](Evans and Honkapohja 2001) provides a comprehensive treatment of the learning approach. See also the surveys (Evans and Honkapohja 1999) and (Marimon 1997).

policy, we in fact do more as we provide general results that are applicable to a wide variety of multivariate linear models. Our applications to monetary policies illustrate how to use the results in an economic framework. We first consider models without lags of endogenous variables for which general theoretical results can be obtained. We then develop stability conditions for models with lags. The latter can be used in numerically calibrated models even though general analytical results are not obtainable.

As our application, we develop stability/instability conditions of non-fundamental REE under learning for versions of the New Keynesian model of monetary policy.[4] Learnable non-fundamental REE can be ruled out if the structural and policy parameters of the model satisfy certain specific conditions that have an economic interpretation and yield important insights about specific interest rate rules. We also assess the plausibility of the (Clarida, Gali, and Gertler 2000) explanation of the Pre-Volcker and Volcker era from the learnability view point.

We remark that expectational errors can naturally arise in practice. The economy might be subject to changes in its basic structure or in the practices of policy makers. The assumption that agents somehow have rational expectations (RE) immediately after such changes is clearly strong and may not be correct empirically. The policy maker would naturally like to adopt policy that is conducive to coordination by the private sector on a desirable equilibrium entailing low inflation and output volatility.

The key general message of our paper is that the monetary policy rule used by the central bank plays a pivotal role in determining the equilibrium selection. Good policy design should ensure that (i) the fundamental equilibrium is stable under adaptive learning and (ii) that possible non-fundamental REE are not stable under learning.

The paper is organized as follows. Section 2 develops a general linear bivariate model without lags, the different types of REE for such models and the conditions for stability under least squares learning for these REE. Section 3 applies the results to the standard New Keynesian model when monetary policy is conducted either through a forward-looking Taylor rule or an optimal discretionary rule proposed by (Clarida, Gali, and Gertler 1999). Section 4 incorporates lagged endogenous variables to the general model of Section 2. Section 5 applies the generalized framework to the issues analyzed by (Clarida, Gali, and Gertler 2000). Conclusions and appendices follow.

2 A General Model Without Lags

Consider a general bivariate linear model

$$x_t = \Omega \hat{E}_t x_{t+1} + \Phi w_t \quad (1)$$

$$w_t = \Psi w_{t-1} + v_t; \quad (2)$$

where $x_t, w_t \in R^2$ are, respectively, the vectors of endogenous and exogenous variables and all constants have been eliminated by centering the variables. The exogenous vari-

[4]Stability of the fundamental REE under adaptive learning is studied by (Bullard and Mitra 2002) for variants of Taylor rules. (Evans and Honkapohja 2003d) analyse the learnability of the fundamental REE for different ways of implementing optimal monetary policy under discretion.

ables follow a stationary vector autoregressive process, so that the eigenvalues of Φ are inside the unit circle. v_t is iid. $\hat{E}_t(.)$ is a general notation for expectations and the same notation without the "^" denotes RE. The limitation to a bivariate model is not crucial, as many results extend to general multivariate frameworks. These will be noted below. For the main part we assume that the 2×2 matrix Ω is invertible, but we will take note of the necessary modifications when Ω is singular.

2.1 Characterization of Non-Fundamental Solutions

2.1.1 Autoregressive Solutions

In this and the next subsection we impose RE, so that $\hat{E}_t x_{t+1} = E_t x_{t+1}$, the mathematical conditional expectation. A very common way to obtain non-fundamental solutions is to represent classes of solutions in terms of arbitrary (unanticipated) innovations to the expectations. Thus let $\eta_{t+1} = x_{t+1} - E_t x_{t+1}$ be any innovation process, so that it satisfies $E_t \eta_{t+1} = 0$ i.e. it is a (vector) martingale difference sequence (MDS). The innovations can depend on extraneous variables and the term "sunspot equilibria" (or sunspot solutions) is then used.

The general class of solutions of (1)-(2) can be written in the form[5]

$$\begin{aligned} x_t &= \Omega^{-1} x_{t-1} - \Omega^{-1}\Gamma w_{t-1} + \eta_t \\ w_t &= \Phi w_{t-1} + v_t \end{aligned} \tag{3}$$

or, introducing the notation $y_t = (x_t', w_t')'$, $u_t = (\eta_t', v_t')'$ in the VAR form

$$y_t = B y_{t-1} + u_t; \text{ where } B = \begin{pmatrix} \Omega^{-1} & -\Omega^{-1}\Gamma \\ 0 & \Phi \end{pmatrix}. \tag{4}$$

Since there are many ways of specifying the innovation process η_t it is evident that in general there are indeterminacies of REE. The only restriction we have on η_t is that it must be a MDS. However, a common further restriction is stationarity of the process (4) and we consider this next.

We diagonalize the coefficient matrix of (4), so that $B = Q\Lambda Q^{-1}$ and introduce the notation $Q^{-1} = (Q^{ij})$. We note that Λ is a diagonal matrix with the eigenvalues of B along its diagonal, i.e. $\Lambda = \{\lambda_1, \ldots, \lambda_4\}$. Since B is block-triangular, the last two eigenvalues are those of Φ and they are inside the unit circle. The remaining two eigenvalues of B (λ_1 and λ_2) are then given by those of Ω^{-1}. If both λ_1 and λ_2 are inside the unit circle, then (3) forms a stationary class of solutions. However, it may also be the case that one or both roots of Ω^{-1} are outside the unit circle. If both roots of Ω^{-1} are outside the unit circle, we have the so-called regular case and only the fundamental solution is stationary and takes the form $y_t = f w_t$, where f satisfies $f = \Omega f \Phi + \Gamma$.

[5] There is a large literature on representing solutions to linear RE models, see e.g. (Broze and Szafarz 1991) or Part III in (Evans and Honkapohja 2001).

If just one of the roots is outside the unit circle, there exist stationary non-fundamental solutions that can be derived by using an extension of the diagonalization technique originally developed in (Blanchard and Kahn 1980). This procedure is normally applied to the original structural model (1)-(2). Since invertibility of Ω has been assumed, the same procedure can equally well be applied to the form (4). The following proposition represents the class of non-fundamental stationary solutions to (1)-(2) in this case.

Proposition 1 Assume without loss of generality (w.l.o.g) that $|\lambda_1| < 1$, $|\lambda_2| > 1$ for the two eigenvalues of Ω^{-1}. The class of stationary autoregressive solutions takes the form

$$\begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} = \begin{pmatrix} \lambda_1 Q^{11} & \lambda_1 Q^{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{1,t-1} \\ x_{2,t-1} \end{pmatrix} - \begin{pmatrix} Q^{13} & Q^{14} \\ Q^{23} & Q^{24} \end{pmatrix} \begin{pmatrix} w_{1,t} \\ w_{2,t} \end{pmatrix}$$
$$+ \begin{pmatrix} \lambda_1 Q^{13} & \lambda_1 Q^{14} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{1,t-1} \\ w_{2,t-1} \end{pmatrix} + \begin{pmatrix} Q^{11} & Q^{12} & Q^{13} & Q^{14} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_{1,t} \\ \eta_{2,t} \\ v_{1,t} \\ v_{2,t} \end{pmatrix}. \quad (5)$$

Proof. See Appendix A.1. ■

Examining (5) it can be seen that the second equation is a linear restriction between the current state and exogenous variables. This limits the degrees of freedom in choosing the components of the innovation η_t as indicated in Appendix A.1.

Two further remarks are worth making at this point. First, if Ω is singular, the representation (4) does not exist. Using an analogous diagonalization procedure on the coefficient matrix of the system

$$\begin{pmatrix} x_t \\ w_t \end{pmatrix} = \begin{pmatrix} \Omega & \Phi a^{-1} \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} x_{t+1} \\ w_{t+1} \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \eta_{t+1} \\ v_{t+1} \end{pmatrix},$$

where $\eta_{t+1} = x_{t+1} - E_t x_{t+1}$, the stationary RE solutions can be obtained; see Chapter 10, Appendix 2 of (Evans and Honkapohja 2001). Second, if the system (1)-(2) is higher-dimensional, the same techniques can be used. However, different classes of stationary solutions may emerge when the procedure is applied to the general solution class (4).

2.1.2 Markov Solutions

The representation of the non-fundamental REE will turn out to be critical in the study of learnability and the standard methodology above does not readily yield all representations of stationary REE to (1)-(2). A different representation of some of the stationary REE can be constructed as follows.[6] These solutions can be derived from the

[6] These kinds of solutions generalize, for a linear framework with AR(1) exogenous variables and shocks, the class of sunspot equilibria introduced in (Chiappori, Geoffard, and Guesnerie 1992).

general form (3) by suitably defining the innovation process η_t, as shown in Appendix A.2.[7]

Suppose that economic agents condition their expectations on a sunspot process s_t, which is an ergodic Markov chain taking values in a finite set $\{1, \ldots, K\}$, $K \geq 2$. We denote its transition matrix by $\Pi = (\pi_{ij})$, where π_{ij} is the probability that the sunspot will be in state j next period if it is in state i in the current period. At time t and with sunspot in state s at that time we consider a solution of the form

$$y_{t,s} = a_s + f w_t, \tag{6}$$

where $y_{t,s}$ denotes the vector of endogenous variables at time t and state s. The intercept vector is thus made dependent on the state of the sunspot process s_t. The following result shows that this kind of equilibria exist:

Proposition 2 There exist Markov RE solutions of the form (6) if at least one eigenvalue of Π is equal to the inverse of an eigenvalue of β, i.e. $|I_4 - \Pi \otimes \beta| = 0$,[8] and where matrix f solves the equation $f = \beta f \Psi + \Phi$.

Proof. Consider solutions of the form (6). Computing the conditional expectation

$$E_{t,s} y_{t+1} = f \Psi w_t + \sum_{i=1}^{K} \pi_{si} a_i, \tag{7}$$

where $E_{t,s}$ denotes the conditional expectation at time t and state s. The structural model (1) with conditioning on the state of the sunspots can be written as $y_{t,s} = \beta E_{t,s} y_{t+1} + \Phi w_t$.

Substituting in the expectations we get

$$y_{t,s} = (\beta f \Psi + \Phi) w_t + \beta \sum_{i=1}^{K} \pi_{si} a_i, \tag{8}$$

so that in the REE the equations

$$f = \beta f \Psi + \Phi \tag{9}$$

$$a_s = \sum_{i=1}^{K} \pi_{si} \beta a_i, \quad s = 1, \ldots, K \tag{10}$$

must hold. Letting $a = (a_1', \ldots, a_K')' \in \mathbb{R}^{2K}$, (10) can be re-written in matrix form as

$$(I - \Pi \otimes \beta) a = 0,$$

[7] (Evans and Honkapohja 2003c) discuss this relation for scalar models without exogenous variables.

[8] The term “resonant frequency” is sometimes used for sunspots that satisfy $|I_4 - \Pi \otimes \beta| = 0$. Note that the equation means that at least one eigenvalue of β is outside the unit circle (so that these solutions can arise in either irregular case where one or both eigenvalues of β are outside the unit circle) since, except for a single eigenvalue of 1, all other eigenvalues of Π are inside the unit circle.

which must have a non-trivial solution by assumption. ■

These Markov sunspot equilibria form a continuum of solutions, since the equilibrium value of $a = (a_1'; \ldots; a_K')'$ is not unique. This non-uniqueness disappears in nonlinear models for which the model (1) is the linearization or log-linearization; see the discussion by (Evans and Honkapohja 2003b) and (Evans and Honkapohja 2003c) in the scalar case.

2.2 Learnability of Non-Fundamental REE

We now consider how the learnability of the non-fundamental equilibria developed above can be analyzed for general forward-looking models (1)-(2). We employ the methodology explicated in (Evans and Honkapohja 2001) as it is by now standard. In this approach the conditions for learnability of REE are given by E-stability conditions.

From the literature it is known that in most cases E-stability provides precisely the conditions of the stability under least-squares (and related) learning schemes. However, this theoretical connection sometimes fails for technical reasons. The main case of failure are the continua of RE solutions in linear models and we are indeed facing this situation here. Simulation studies suggest that the connection between E-stability and convergence of least squares learning does hold for solution continua.[9] We note that E-stability is also sometimes interpreted as a highly stylized learning process; see (Evans 1989).

With these remarks in mind we employ the E-stability criterion in our analysis of learnability of the REE. The analysis of E-stability of the different types of REE for the general model (1)-(2) can generally be developed as follows.

2.2.1 E-Instability of Autoregressive Solutions

We begin with the classes of REE taking the form (3) or (5). The analysis of E-stability begins with the (in general non-rational) perceptions of the agents. We thus introduce the perceived law of motion (PLM)

$$x_t = a + bx_{t-1} + cw_{t-1} + d\varepsilon_t + ev_t; \qquad (11)$$

where a, b, c, d and e are parameter matrices or vectors of appropriate dimensions. Note that this form of the PLM is the same as (3) and (5), but with parameter values that are in general different from any REE. Note also that we have allowed for a possible intercept in the PLM.

E-stability of REE of the form (3) or (5) is sensitive with respect to the information available in the updating of the parameters of the PLM. We first note that non-fundamental equilibria cannot be E-stable if the period t values of the endogenous variables are included in the information set when agents update the PLM parameters,

[9] See (Evans and Honkapohja 1994a) and Part III of (Evans and Honkapohja 2001) for a discussion of these questions and for further references. The simulations use univariate models, but there appears to be no reason why the situation would be different for multivariate models.

see Chapter 10 of (Evans and Honkapohja 2001). Here we outline the intuition behind this result. Under this assumption, we get from (11)

$$\hat{E}_t x_{t+1} = a + bx_t + c\rho w_t.$$

The key to the instability result is that the future forecast, $\hat{E}_t x_{t+1}$, is independent of the sunspot η_t and hence, the ALM is also independent of this term. Put differently, the agents' belief in the sunspot η_t with coefficient d is invalidated by the actual data that is generated as a result of this belief. Consequently, as agents accumulate more data over time, they learn not to believe in this sunspot and converge to an equilibrium with $d = 0$.

Henceforth, we assume that, when making forecasts at time t, agents can only observe the endogenous variables at time $t-1$ but they observe the values of the exogenous variables (including shocks) at time t. This is natural in economic contexts since agents rarely observe contemporaneously dated endogenous variables while making their forecasts. Agents make forecasts using the PLM (11) with given values of the parameters, so that these forecasts are given by

$$\begin{aligned} \hat{E}_t x_{t+1} &= a + b\hat{E}_t x_t + c\rho w_{t-1} + cv_t \\ &= a + ba + b^2 x_{t-1} + (bc + c\rho)w_{t-1} + bd\eta_t + (be + c)v_t. \end{aligned} \qquad (12)$$

Substituting these forecasts into (1) leads to the actual law of motion (ALM) of the form

$$x_t = \beta[a + ba + b^2 x_{t-1} + (bc + c\rho)w_{t-1} + bd\eta_t + (be + c)v_t] + \delta\rho w_{t-1} + \delta v_t. \qquad (13)$$

The ALM describes the temporary equilibrium of the economy when agents use the PLM with the specified parameter values when forming expectations. Note that unlike the previous case (when t dated endogenous variables were included in the information set), the sunspot η_t affects $\hat{E}_t x_{t+1}$ as in (12) and hence x_t as in (13). This means that sunspots can potentially be E-stable now.

We have obtained a mapping

$$(a; b; c; d; e) \to T(a; b; c; d; e)$$

from the PLM to ALM, where

$$T(a; b; c; d; e) = (\beta(I + b)a; \beta b^2; \beta(bc + c\rho) + \delta\rho; \beta bd; \beta(be + c) + \delta).$$

The different REE are fixed points of the T mapping and satisfy the matrix equations

$$a = \beta(I + b)a \qquad (14)$$

$$b = \beta b^2 \qquad (15)$$

$$c = \beta(bc + c\rho) + \delta\rho \qquad (16)$$

$$d = \beta bd \qquad (17)$$

$$e = \beta(be + c) + \delta. \qquad (18)$$

It can be seen that the equation for matrix b is a quadratic matrix equation. Clearly, $b = \beta^{-1}$ solves this equation, but in general it has other solutions. Some of the solutions can be singular matrices and this possibility will be illustrated below.

Given a solution $\bar{b}$, equations (14) and (16) generically uniquely determine a and c ($\bar{a}$ and $\bar{c}$): Given $\bar{b}, \bar{c}$, (18) solves e uniquely. For sunspot equilibria, equation (17) has non-trivial multiple solutions for d when, given $\bar{b}$, the matrix $I - \beta\bar{b}$ is singular. This happens e.g. when $b = \beta^{-1}$.

Letting $\xi' = (a, b, c, d, e)$, E-stability of a fixed point $\bar{\xi}' = (\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e})$ is defined using the ordinary differential equation

$$\frac{d\xi}{d\tau} = T(\xi) - \xi. \tag{19}$$

A fixed point $\bar{\xi}$ of $T(\xi)$ is E-stable if it is locally asymptotically stable under (19).[10] Formally, this differential equation describes partial adjustment in continuous (artificial) time τ between the PLM that the agents use in forecasting and the actual outcome of the economy under these forecast functions. Since we will analyze continua of RE solutions, E-stability of a class of equilibria must also be defined and we follow (Evans and Honkapohja 2001), p. 245. Let $S(\bar{b})$ be the set of fixed points $\phi = (a, b, c, d, e)$ of $T(.)$ when $b = \bar{b}$. The class $S(\bar{b})$ is E-stable if, for some neighborhood N of $S(\bar{b})$ the solution $\phi(\tau)$ of (19) for any initial condition $\phi_0 \in N$ converges to ϕ_∞, where $\phi_\infty \in S(\bar{b})$.

Intuitively, E-stability of a solution depends on the strength of the feedback from expectations of the endogenous variables, $\hat{E}_t x_{t+1}$, to their actual values x_t in the model (1). E-instability of a solution results when an initial shift in $\hat{E}_t x_{t+1}$ away from the REE leads to actual changes in x_t which deviate further (in some metric) than the initial change in $\hat{E}_t x_{t+1}$ from equilibrium, thereby inducing further divergent changes in $\hat{E}_t x_{t+1}$ under learning and divergence from the REE. In contrast, E-stability results when a shift in $\hat{E}_t x_{t+1}$ leads to changes in actual x_t that are closer to the REE than the initial shift in $\hat{E}_t x_{t+1}$, which implies gradual convergence back to the REE.

To derive the E-stability and E-instability conditions we linearize (19). Since the system is matrix-valued, it must be vectorized. We use standard results from matrix algebra and analysis of multivariate linear models, see Chapter 10 of (Evans and Honkapohja 2001), to obtain the coefficient matrices of the linearized and vectorized form of (19). This yields the necessary E-stability condition that the real parts of the eigenvalues of the following matrices

$$\begin{aligned} DT_a(\bar{a}, \bar{b}) &= \beta(I + \bar{b}) \\ DT_b(\bar{b}) &= \bar{b}' \otimes \beta + I \otimes \beta\bar{b} \\ DT_c(\bar{b}, \bar{c}) &= \rho' \otimes \beta + I \otimes \beta\bar{b} \\ DT_d(\bar{b}) &= \beta\bar{b} \end{aligned} \tag{20}$$

[10] A fixed point $\bar{\xi}$ of (19) is said to be stable if for every neighborhood V of $\bar{\xi}$, there exists a neighborhood $V_1 \subset V$ such that every solution $\xi(\tau)$ with $\xi(0) \in V_1$ lies in V for all $\tau > 0$. If, in addition, V_1 can be chosen so that $\xi(\tau) \to \bar{\xi}$ as $\tau \to \infty$, then $\bar{\xi}$ is said to locally asymptotically stable; see e.g. (Guckenheimer and Holmes 1983), pp. 3-4.

must have real parts less than one. The sufficient condition for E-instability is that at least one eigenvalues of these matrices has a real part greater than one.

We have the result that the class of non-fundamental REE (3) for the structural model (1)-(2) is not E-stable:[11]

Proposition 3 The solution class (3) is not E-stable.

Proof. Using (20), if we evaluate $DT_b(b)$ at $b = \Gamma^{-1}$; we get $(\Gamma^{-1})' \otimes \Gamma + I_4$; where I_4 denotes the 4×4 identity matrix. Using the properties of the eigenvalues of the Kronecker product of two matrices, we observe that the eigenvalues of $DT_b(\Gamma^{-1})$ include an unstable root (with value 2), which proves the result. ■

Remark: This proposition also holds in higher dimensional forward-looking models.

This proposition has a limitation. In the analysis of E-stability we have not imposed the requirement that the RE solutions and the possible PLM be stationary. Of course, if both eigenvalues of Γ are outside the unit circle, then (3) forms a class of stationary sunspot solutions (SSE) and the requirement is satisfied. If only one eigenvalue of Γ is outside the unit circle, then (3) does not form a class of stationary sunspot solutions. One class of SSEs are given by the class (5) in Proposition 1 and we now consider its E-stability.

First we note that, since E-stability is local concept, all non-rational PLM's sufficiently near these REE must be stationary. We have the following result.

Proposition 4 The class of stationary REE (5) is not E-stable.

Proof. We will show that instability arises from the eigenvalues of the matrix $DT_b(b)$ evaluated at this REE. We first need to compute the solution $\bar{b}$ which is given by

$$\bar{b} = \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 Q^{11} & \lambda_1 Q^{12} \\ 0 & 0 \end{pmatrix}. \tag{21}$$

The eigenvalues of $\bar{b}$ are 0 and λ_1 so that this solution is stationary. A necessary condition for $\bar{b}$ to be E-stable is that the eigenvalues of the matrix

$$DT_b(\bar{b}) = \bar{b}' \otimes \Gamma + I \otimes \Gamma\bar{b}$$

have real parts less than 1. However, one of its eigenvalues is always 2, which proves the result. Appendix A.3 outlines a Mathematica routine to calculate the roots of $DT_b(\bar{b})$. ■

Intuitively, E-instability of the autoregressive non-fundamental equilibria, the solution classes (3) and (5), arises from strong effects of expectations on the current values of the endogenous variables. These solutions are of higher order than the MSV solution and, in particular, include feedback from the lagged endogenous variables. In either case,

[11] (Evans 1989) analyzed E-instability of the solution class (3) for univariate forward-looking models.

past values of x_t have a strong influence on expectations $\hat{E}_t x_{t+1}$, (12), and hence, current x_t in (13) leading to instability; see the proofs of Propositions 3 and 4. For example, with solution class (3), $\bar{b} = \Omega^{-1}$; $DT_b(b)$ has an eigenvalue of 2. The strong feedback means that changes in $\hat{E}_t x_{t+1}$ lead to changes in actual x_t which are (in some metric) even larger and lead to a divergence from these types of non-fundamental equilibria under learning dynamics.[12]

We remark that in the formulation of learning, we have endowed agents with knowledge of the form of the REE, compare (11) with (3) or (5). If agents are not able to converge to the REE under such favorable conditions, then one cannot expect convergence when they have less a priori knowledge of the REE. Consequently, these non-fundamental equilibria would continue to be E-unstable under more general conditions.[13]

Summing up, Propositions 3 and 4 show that the autoregressive classes (3) and (5) of non-fundamental REE in purely forward-looking models are E-unstable. Proposition 4 is currently limited to bivariate models, but we conjecture that it also holds generally.

2.2.2 E-Stability of Markov Sunspots

Finally, we analyze E-stability of the Markov sunspot REE of the form (6).[14] Thus assume that agents have PLM of that form but a_s and f do not take the REE values given by equations (9) and (10). The right-hand sides of (9) and (10) define the $T-$mapping used in the analysis of E-stability in the standard way. Thus denote

$$T_a(a) = (\Pi \otimes \Omega)a \quad (22)$$
$$T_f(f) = \Omega f \Psi + \Phi \quad (23)$$

in matrix form. Introducing the notation $\xi = (a; f)$; $T(\xi) = (T_a(a); T_f(f))$; E-stability is defined as usual by the differential equation

$$\frac{d\xi}{d\tau} = T(\xi) - \xi.$$

For these non-fundamental equilibria, a sufficient condition for E-instability is:[15]

Proposition 5 The class of sunspot equilibria of the form (6) are not E-stable if Ω has an eigenvalue with real part > 1.

Proof. Consider the component $T_a(a) = (\Pi \otimes \Omega)a$ of the $T-$mapping constructed in the proof of Proposition 2. Its eigenvalues are the products of the eigenvalues of Π

[12] We will see below that this feature does not arise with MSV or the Markov sunspots since trivially $\bar{b} = 0$ in these cases. In addition, note that one can't expect any necessary link between Propositions 3 and 4: E-stability is a local concept and the value of $\bar{b}$ is different for the two solution classes.

[13] In the parlance of the learning literature, if an REE is not weakly E-stable, then it can not be strongly E-stable; see (Evans and Honkapohja 2001) p. 42 for an intuitive discussion.

[14] E-stability of the Markov sunspots for univariate models without exogenous shocks was considered in (Evans and Honkapohja 1994b), (Evans and Honkapohja 2003c) and (Evans and Honkapohja 2003b).

[15] This result was first obtained in (Evans and Honkapohja 1994b) for scalar models without shocks.

and Ω. Since 1 is an eigenvalue of the probability matrix Π, the matrix $\Pi \otimes \Omega$ has an eigenvalue with real part greater than one. ■

What about the possibility of E-stable Markov sunspots? For the non-stochastic scalar model (where in our notation $\Omega < -1$) and a two-state sunspot process, (Evans and Honkapohja 2003c) recently discovered that there are E-stable Markov sunspot solutions. Here we extend that result for the multivariate stochastic setup (1)-(2):

Proposition 6 Assume that (i) all eigenvalues of $\Psi' \otimes \Omega$ have real parts < 1 and that (ii) with the exception of a single eigenvalue equal to 1 (which exists by Proposition 2) the other eigenvalues of $\Pi \otimes \Omega$ have real parts < 1. Then the class of Markov sunspot equilibria are E-stable.

Proof. We first vectorize the matrix-valued differential equation

$$\frac{df}{d\tau} = \Omega f \Psi + \Phi - f.$$

This yields

$$\frac{d(\mathrm{vec} f)}{d\tau} = (\Psi' \otimes \Omega - I)\mathrm{vec} f + \mathrm{vec}\Phi,$$

which is stable by assumption (i).

Next consider the (linear) differential equation for a. Its coefficient matrix $\Pi \otimes \Omega - I$ has a single eigenvalue equal to zero while the others are, by hypothesis, stable. The mathematical lemma in Appendix A.4 shows that for such systems we have convergence to the set of equilibrium points. ■

The different E-stability properties of the Markov sunspot and autoregressive solutions may seem surprising, since an appropriate specification of η_t in the latter gives the same RE solution as the former. However, this is reconciled by observing that the parametric form of the PLM matters for the learnability properties; see (Evans and Honkapohja 2003c) for a further discussion.

Intuition for Proposition 6 is developed below in Section 3.1. Propositions 2, 5 and 6 show the following corollary:

Corollary 7 There may exist E-stable SSEs when the parameter matrix Ω has a real eigenvalue < -1.

This is accomplished by selecting the transition matrix Π so that (i) and (ii) of Proposition 6 can be met.[16]

As was pointed out in the introduction, it is important to consider whether an equilibrium is robust to small expectational errors and learning mechanisms to correct them. When the model exhibits indeterminacy, stability under learning (or E-stability) can provide a selection criterion between the fundamental and non-fundamental REE. In the next section we will apply these general results on E-stability of non-fundamental REE to a standard New Keynesian model of monetary policy.

[16] If Ω has more than one real eigenvalue < -1, one selects Π so that the inverse of just one of these eigenvalues is an eigenvalue of Π.

3 Applications to Monetary Policy

We conduct the analysis using the framework in Section 2 of (Clarida, Gali, and Gertler 1999). The structural model consists of two equations:

$$z_t = -\varphi(i_t - \hat{E}_t\pi_{t+1}) + \hat{E}_t z_{t+1} + g_t; \quad (24)$$
$$\pi_t = \lambda z_t + \beta\hat{E}_t\pi_{t+1} + u_t; \quad (25)$$

where z_t is the "output gap" i.e. the difference between actual and potential output, π_t is the inflation rate, i.e. the proportional rate of change in the price level from $t-1$ to t and i_t is the nominal interest rate. $\hat{E}_t\pi_{t+1}$ and $\hat{E}_t z_{t+1}$ denote private sector expectations of inflation and output gap next period. All the parameters in (24) and (25) are positive. $0 < \beta < 1$ is the discount rate of the representative firm.

(24) is a dynamic "IS" curve that can be derived from the Euler equation associated with the household's savings decision. (25) is a "new Phillips curve" that can be derived from optimal pricing decisions of monopolistically competitive firms facing constraints on the frequency of future price changes.

g_t and u_t denote observable shocks following first order autoregressive processes

$$g_t = \mu g_{t-1} + \tilde{g}_t; \quad (26)$$
$$u_t = \rho u_{t-1} + \tilde{u}_t; \quad (27)$$

where $0 < |\mu| < 1; 0 < |\rho| < 1$ and $\tilde{g}_t \sim iid(0; \sigma_g^2); \tilde{u}_t \sim iid(0; \sigma_u^2)$. g_t represents shocks to government purchases as well as shocks to potential GDP. u_t represents cost push shocks to marginal costs.

Monetary policy is conducted by means of control of the nominal interest rate i_t. A number of different types of control have been analyzed in the literature and consider two well-known interest rate rules.

3.1 Forward-Looking Taylor Rules

The nominal interest rate is assumed to be adjusted in accordance with expectations of output gap and inflation next period. For simplicity, we assume that the expectations of private agents and policy makers are identical.[17] Then

$$i_t = \chi_\pi \hat{E}_t\pi_{t+1} + \chi_z \hat{E}_t z_{t+1} \quad (28)$$

and the structural model becomes

$$\begin{pmatrix} z_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1-\varphi\chi_z & \varphi(1-\chi_\pi) \\ \lambda(1-\varphi\chi_z) & \beta+\lambda\varphi(1-\chi_\pi) \end{pmatrix}\begin{pmatrix} \hat{E}_t z_{t+1} \\ \hat{E}_t\pi_{t+1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} g_t \\ u_t \end{pmatrix}. \quad (29)$$

[17] See (Honkapohja and Mitra 2003) for an analysis of the model when the central bank uses its own internal forecasts in interest rate setting.

This model may be determinate or indeterminate. (Bullard and Mitra 2002) show that the conditions for determinacy are $\chi_x < \varphi^{-1}(1+\beta^{-1})$, $\lambda(\chi_\pi - 1) + (1+\beta)\chi_x < 2\varphi^{-1}(1+\beta)$ and $\lambda(\chi_\pi - 1) + (1-\beta)\chi_x > 0$. These conditions show that central bank policy may easily lead to indeterminacies, which caused (Bernanke and Woodford 1997) to argue against them. However, this leaves open the question as to whether these sunspots may be learnable by private agents, especially for plausible values of parameters. If these sunspots are either never learnable or are learnable only under very implausible parameter specifications while the MSV solution turns out to be stable for plausible values, then arguably the policy can still lead agents to coordinate on the MSV solution.[18]

We, therefore, turn to the implications for E-stability results on non-fundamental equilibria associated with (28). We know that non-fundamental equilibria of the form (3) and (5) are never E-stable. One can show that, depending on the structural parameters and policy coefficients, there may exist eigenvalues which are more than 1 or less than -1 in the indeterminate case. Proposition 5 implies that non-fundamental equilibria of the form (6) are not E-stable when the “irregular” eigenvalues are greater than 1.

Nevertheless, it is easy to show that a set of sufficient conditions for Ω to have one eigenvalue less than -1 (and the other in the interval $(-1, 1)$) are $\chi_\pi > 1$ and $\chi_x \geq 2\varphi^{-1}$. Proposition 6 and Corollary 7, hence, show that E-stable sunspots do exist and in fact they do so for plausible values of parameters, e.g. when $\varphi^{-1} = .157$, as in (Woodford 1999). The learning perspective, therefore, strengthens the worries of (Bernanke and Woodford 1997).

A theme that we will elaborate further in Section 6 is the connection between E-stability and the “Taylor principle”, see Chapter 4 of (Woodford 2003b) for a definition. Intuitively, the Taylor principle means that nominal interest rates rise by more than the increase in the inflation rate in the long-run. (Bullard and Mitra 2002) showed earlier this connection for the fundamental REE/MSV solution: rules fulfilling the Taylor principle are learnable and rules violating the principle are unlearnable. Our results indicate that the connection partly extends to the set of non-fundamental REE under the forward-looking Taylor rule (28). Policies violating the Taylor principle lead to indeterminacy and the MSV as well as all of the non-fundamental REE are then unlearnable. However, when the policy (28) conforms with the Taylor principle but also implies indeterminacy, then both fundamental and non-fundamental Markov REE are learnable.

Some intuition for the results on Markov SSEs can be developed as follows. The MSV equilibrium is of the form $y_t = a + fw_t$, which is formally a special case of the solution (6) with $a_s = a$ for all s. In (Bullard and Mitra 2002) the constant term was the key to E-stability/instability of the MSV solution and the mapping from the PLM to the ALM for the constant term in their case was $a \to \Omega a$, where Ω is the coefficient matrix of the expectations term in (29). E-stability requires the eigenvalues of Ω to have real parts less than one, which is equivalent to the Taylor principle.

For Markov SSEs it is again the mapping of the constant term(s) which determines E-stability; see Proposition 6. The constant term takes different values depending on the state of the sunspot and the conditional expectation (7) (which are affected by the

[18] This is the crux of the argument in (McCallum 2001a), (McCallum 2001b) and (McCallum 2003).

probabilities in Π) influence the model in the way shown in (8). The T mapping for the sunspot-state dependent constant term is (22) and E-stability requires the eigenvalues of $\Pi \otimes \Omega$ to have real parts less than one. This generalizes the E-stability condition for the MSV solution: $\Pi \otimes \Omega$ takes the role of Ω. Since one is always an eigenvalue of Π, instability obtains as soon as Ω has an eigenvalue with real part more than one, which means a violation of the Taylor principle. However, even though the Taylor principle continues to be necessary for E-stability of Markov SSEs, it is no longer sufficient. To simplify, let us focus on the case $K = 2$: If the eigenvalues of Ω are denoted ω_1 and ω_2, then for the existence of these sunspots, Π must have eigenvalues 1 and ω_1^{-1} (w.l.o.g.) by Proposition 2. The eigenvalues of $\Pi \otimes \Omega$ are then $1; \omega_1, \omega_2$; and $\omega_2\omega_1^{-1}$. The Taylor principle merely ensures the real parts of $\omega_1$and ω_2 to be less than 1 but not that of $\omega_2\omega_1^{-1}$: In fact, one can show (we omit the details for brevity) that the Markov sunspots can be unstable when the response to output χ_z in the rule (28) is small enough (in particular, when $\chi_z < \varphi^{-1}$): However, if this response becomes large ($\chi_z > \varphi^{-1}$), these sunspots are E-stable. Our perspective, therefore, supports a modest reaction to output since it rules out learnable sunspots.[19]

3.2 Optimal Monetary Policy Under Discretion

Optimal monetary policies under discretion can also lead to purely forward-looking structures of the form (1)-(2). Postulating a standard quadratic objective function[20]

$$\min \frac{1}{2} E_t \left\{ \sum_{i=0}^{\infty} \beta^i \left[\alpha z_{t+i}^2 + \pi_{t+i}^2 \right] \right\}; \tag{30}$$

we consider optimal monetary policy under discretion. In the fundamental equilibrium optimal monetary policy without commitment can be characterized in several different ways. (Clarida, Gali, and Gertler 1999) show the formula

$$i_t = \left(1 + \frac{(1-\rho)\lambda}{\rho\alpha\varphi}\right) E_t \pi_{t+1} + \varphi^{-1} g_t; \tag{31}$$

for the optimal interest rate under RE. We can think of (31) as a specified interest rate rule and consider the resulting structural model. It takes the form

$$\begin{pmatrix} z_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1 & -(1-\rho)\lambda/\rho\alpha \\ \lambda & \beta - (1-\rho)\lambda^2/\rho\alpha \end{pmatrix} \begin{pmatrix} \hat{E}_t z_{t+1} \\ \hat{E}_t \pi_{t+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_t.$$

[19] This will also lead to learnability of the MSV solution even when indeterminacy obtains. We note that relatively modest responses, particularly to the output gap, are also supported in the very different model of (Christiano and Gust 1999). Similarly, (Orphanides 2003) argues for prudent policies owing to the difficulties in measuring the output gap. Existence of parameter uncertainty can also support a less activist optimal policy for a central bank, see (Wieland 1998).

[20] α is the relative weight for output deviations and β is the discount rate. The policy maker is assumed to discount the future at the same rate as the private sector. Allowing for a deviation of socially optimal output from potential output and a non-zero target for the inflation rate would not affect the results.

(Evans and Honkapohja 2003d) point out that in this case the model is either determinate or indeterminate, depending on the values of structural parameters and ρ. If $-1 < \rho < 0$, there is necessarily indeterminacy since in this case there are two positive eigenvalues with one exceeding 1. On the other hand, when $0 < \rho < 1$, indeterminacy obtains if $\rho < \lambda^2[\lambda^2 + 2\alpha(1+\beta)]^{-1}$ since in this case the characteristic polynomial has a root less than -1 (the other being between -1 and 0).

It follows that the non-fundamental REE are not learnable if the autoregression parameters of the cost push shock satisfies $-1 < \rho < 0$. However, in the empirically plausible case $0 < \rho < 1$ there are situations of indeterminacy and by Proposition 6, E-stable sunspots are possible. The same general intuition based on the eigenvalues holds as in the case of the Taylor rule in the preceding section.

Most central banks have some concerns for output even when they pursue a policy of inflation targeting. These banks are often said to practice flexible inflation targeting; see (Svensson 1999). This means a positive value of α in terms of (30). On the other hand, a policy which aims to only target inflation is called strict inflation targeting and this corresponds to $\alpha = 0$. When α is close to 0, the policy is almost certain to lead to indeterminacies when $\rho > 0$ (by our previous arguments). A policy of strict inflation targeting is typically believed to increase the volatility of output even when it leads to less volatility in inflation. The usual reasoning presumes that the economy is in the MSV solution. We provide an additional reason to avoid this policy: it may lead the private sector to believe in sunspots entailing a (potentially) large volatility in both inflation and output. Furthermore, the private sector may even learn to converge on these sunspots. Our perspective, therefore, provides additional support for a policy of flexible inflation targeting.

4 A General Model with Lags

In this section we discuss the stability of stationary sunspot equilibria in models with lags. Some recent models of monetary policy lead to such formulations. We consider the general class of models

$$x_t = \beta \hat{E}_t x_{t+1} + \delta x_{t-1} + \Phi w_t \tag{32}$$

$$w_t = \Psi w_{t-1} + v_t, \tag{33}$$

where x_t, w_t are, respectively, the vectors of endogenous and exogenous variables. The exogenous variables follow a stationary VAR, so that the eigenvalues of Ψ are inside the unit circle. v_t is iid.

4.1 Characterization of Non-Fundamental Solutions

In this subsection we impose RE, so that $\hat{E}_t x_{t+1} = E_t x_{t+1}$. Let $\eta_{t+1} = x_{t+1} - E_t x_{t+1}$ be any innovation process, so that it satisfies $E_t \eta_{t+1} = 0$. For the main part we assume that the matrix β is invertible, but we will also encounter the case where β is singular and will point out the modifications to the technique.

The general solution of (32)-(33) can be written in the form

$$
\begin{aligned}
x_t &= \beta^{-1} x_{t-1} - \beta^{-1}\delta x_{t-2} - \beta^{-1}\kappa w_{t-1} + \varepsilon_t \\
w_t &= \rho w_{t-1} + v_t
\end{aligned} \tag{34}
$$

or, introducing the notation $y_t = (x_t', x_{t-1}', w_t')'$, $u_t = (\varepsilon_t', v_t')'$, in the form

$$
y_t = B_1 y_{t-1} + L u_t, \text{ where } B_1 = \begin{pmatrix} \beta^{-1} & -\beta^{-1}\delta & -\beta^{-1}\kappa \\ I & 0 & 0 \\ 0 & 0 & \rho \end{pmatrix}; L = \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix}. \tag{35}
$$

Note that this formulation does not assume invertibility of δ. Since there are many ways for specifying the innovation process ε_t, there can be indeterminacies of REE as before. A very common further restriction on (35) is stationarity of the process. The general characterization (35) will form a stationary class of solutions if all eigenvalues of B_1 are inside the unit circle. However, it may also be the case that one or more roots of B_1 are outside the unit circle and one can obtain stationary solutions using diagonalization techniques similar to the ones in Section 2.

4.2 Learnability of Non-Fundamental REE

Using the concept of E-stability, learnability of the non-fundamental REE can be analyzed as before. Thus only a summary discussion will be given.

We proceed from the structural model (32)-(33) and begin with the PLM

$$
x_t = a + b_1 x_{t-1} + b_2 x_{t-2} + c w_{t-1} + d\varepsilon_t + e v_t,
$$

where a, b_1, b_2, c, d and e are parameter matrices or vectors of appropriate dimensions. The form of the PLM is the same as (34) but with different coefficients. Assuming that agents observe the time t values of the exogenous variables and shocks but not of the endogenous variables to form forecasts[21], the PLM yields an ALM and the mapping from the PLM to ALM, $(a, b_1, b_2, c, d, e) \to T(a, b_1, b_2, c, d, e)$, which is

$$
\begin{aligned}
T(a, b_1, b_2, c, d, e) = \; & (\beta(I + b_1)a, \beta(b_1^2 + b_2) + \delta, \\
& \beta b_1 b_2, \beta(b_1 c + c\rho) + \kappa\rho, \beta b_1 d, \beta(b_1 e + c) + \kappa).
\end{aligned}
$$

[21] As in the case of no lags, the equilibria will be E-unstable if the information set includes period t values of the endogenous variables. The intuition remains the same.

The different REE are fixed points of the T mapping and satisfy the equations

$$\begin{aligned} a &= \beta(I + b_1)a & (36)\\ b_1 &= \beta(b_1^2 + b_2) + \delta & (37)\\ b_2 &= \beta b_1 b_2 & (38)\\ c &= \beta(b_1 c + c\rho) + \kappa\rho & (39)\\ d &= \beta b_1 d & (40)\\ e &= \beta(b_1 e + c) + \kappa. & (41) \end{aligned}$$

It can be seen that the equation for matrices b_1 and b_2 form an independent sub-system and involve matrix quadratic equations. Clearly, the values $b_1 = \beta^{-1}$ and $b_2 = -\beta^{-1}\delta$ in (34) solve this sub-system, but in general there are other solutions. Some of the solutions can be singular matrices.

Given a solution $\bar{b}_1, \bar{b}_2$, equations (36) and (39) generically uniquely determine a and c ($\bar{a}$ and $\bar{c}$). Given $\bar{b}_1$ and $\bar{c}$, (41) uniquely determines e. For sunspot equilibria the matrix $I - \beta\bar{b}_1$ must be singular (which occurs e.g. when $\bar{b}_1 = \beta^{-1}$), in which case the equation for d has nontrivial solutions and sunspot equilibria exist.

E-stability of a fixed point is defined by means of the ordinary differential equation

$$\frac{d}{d\tau}(a, b_1, b_2, c, d, e) = T(a, b_1, b_2, c, d, e) - (a, b_1, b_2, c, d, e). \quad (42)$$

To derive the E-stability and instability conditions we linearize (42). The necessary E-stability conditions are that the real parts of all the eigenvalues of the following matrices

$$\begin{aligned} &\beta(I + \bar{b}_1),\\ &\bar{a}' \otimes \beta + I \otimes \beta\bar{b}_1,\\ &\beta\bar{b}_1,\\ &\begin{pmatrix} \bar{b}_1' \otimes \beta + I \otimes \beta\bar{b}_1 & I \\ \bar{b}_2' \otimes \beta & I \otimes \beta\bar{b}_1 \end{pmatrix}. \end{aligned} \quad (43)$$

have real parts less than one. On the other hand, a solution or a class of solutions is E-unstable if any of the eigenvalues of these matrices has a real part exceeding one.

In this case, clear-cut theoretical results for E-instability are generally not available since, for example, the eigenvalues of (43), when evaluated at $b_1 = \beta^{-1}$, $b_2 = -\beta^{-1}\delta$, depend on δ. Nevertheless, these conditions can readily be applied to numerically specified models as will be illustrated below.[22]

5 Application to Clarida, Gali, and Gertler (2000)

We now consider the model analyzed in Section 4 of (Clarida, Gali, and Gertler 2000), which is similar to the one considered in Section 3. The structural model contains the

[22] (Evans and McGough 2003) have recently analyzed numerically different variants of models of monetary policy with lags, including a further analysis of the model of (Clarida, Gali, and Gertler 2000).

IS curve (24) but (25) is replaced by a slightly modified equation, namely,

$$\pi_t = \lambda z_t + \beta \hat{E}_t \pi_{t+1} - \lambda u_t, \tag{44}$$

where the parameters are the same as in Section 3 and the shocks g_t and u_t continue to follow the processes (26) and (27). (Clarida, Gali, and Gertler 2000) use an interest rate rule of the form

$$i_t = \theta i_{t-1} + (1-\theta)\chi_\pi \hat{E}_t \pi_{t+1} + (1-\theta)\chi_z z_t, \tag{45}$$

which has an inertial component captured by θ and reacts to the contemporaneous output gap and the forecast of future inflation.

Plugging this rule into (24) and (44) yields the reduced from

$$\begin{pmatrix} z_t \\ \pi_t \\ i_t \end{pmatrix} = \Omega \begin{pmatrix} \hat{E}_t z_{t+1} \\ \hat{E}_t \pi_{t+1} \\ \hat{E}_t i_{t+1} \end{pmatrix} + \delta \begin{pmatrix} z_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \kappa \begin{pmatrix} g_t \\ u_t \end{pmatrix}, \text{ where} \tag{46}$$

$$\begin{aligned}
\Omega &= \begin{pmatrix} k_0 & k_0\varphi\{1-(1-\theta)\chi_\pi\} & 0 \\ \lambda k_0 & \beta + \lambda k_0 \varphi\{1-(1-\theta)\chi_\pi\} & 0 \\ k_0(1-\theta)\chi_z & k_0(1-\theta)(\varphi\chi_z + \chi_\pi) & 0 \end{pmatrix}; \\
\delta &= \begin{pmatrix} 0 & 0 & -k_0\varphi\theta \\ 0 & 0 & -\lambda k_0\varphi\theta \\ 0 & 0 & k_0\theta \end{pmatrix}; \; \kappa = \begin{pmatrix} k_0 & 0 \\ \lambda k_0 & -\lambda \\ k_0(1-\theta)\chi_z & 0 \end{pmatrix}; \\
k_0 &= \{1 + (1-\theta)\varphi\chi_z\}^{-1}.
\end{aligned} \tag{47}$$

It is possible to show that indeterminacies can arise in this model. This model fits the general framework (32)-(33), where x_t is now three-dimensional. In this case neither δ nor Ω are invertible and for computing the indeterminate equilibria we need to apply the diagonalization technique directly on (46) rather than the autoregressive form used previously (See e.g. Chapter 10 of (Evans and Honkapohja 2001).) The E-stability analysis is not affected.

Define the free variables as $x_t^1 = x_t = (z_t, \pi_t, i_t)'$ and the predetermined variables as $x_t^2 = (i_{t-1}, w_{1t}, w_{2t})'$ where $w_{1t} = g_t$ and $w_{2t} = u_t$. The technique starts from the following general form ($e_t = (g_t, u_t)'$ below)

$$x_t^1 = B_1 E_t x_{t+1}^1 + C x_t^2, \tag{48}$$

$$x_t^2 = R x_{t-1}^1 + S x_{t-1}^2 + \kappa_1 e_t, \tag{49}$$

where in our case we have $B_1 = \Omega$,

$$C = \begin{pmatrix} -k_0\varphi\theta & k_0 & 0 \\ -k_0\lambda\varphi\theta & k_0\lambda & -\lambda \\ k_0\theta & k_0(1-\theta)\chi_z & 0 \end{pmatrix};$$

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \rho \end{pmatrix}; \kappa_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

and k_0 is defined in (47). Having put the model in this form we compute the matrix J:

$$J = \begin{bmatrix} I & -C \\ R & S \end{bmatrix}^{-1} \begin{bmatrix} \Omega & 0 \\ 0 & I \end{bmatrix} \tag{50}$$

(see Chapter 10, Appendix 2 of (Evans and Honkapohja 2001) for the details). Equilibrium will be unique if exactly 3 (of the 6) eigenvalues of J are inside the unit circle, while it will be indeterminate if fewer than 3 eigenvalues are inside the unit circle.

We now compute and examine the learnability of indeterminate equilibria in (Clarida, Gali, and Gertler 2000). They have suggested that monetary policy in the pre-Volcker era (i.e., $1960 - 1979$) led the economy to stationary sunspot equilibria. The calibrated parameter values they use are $\varphi = 1; \lambda = .3; \beta = .99; \theta = .68; \psi_z = .27; \mu = \rho = .9$: ψ_π was consistently found to be less than one in this period and is the cause for indeterminacy. If we use the baseline estimate of $\psi_\pi = 0.83$ in Table 2 of their paper, we find that exactly 2 eigenvalues of J are inside the unit circle.

Appendix A.5 shows that the final solution for $x_t = (z_t, \pi_t, i_t)'$ is a process of the form (34) with the corresponding solutions for $\tilde{b}_1, \tilde{b}_2$ given by (60) and (61) in Appendix A.5. This (sunspot) solution will be stationary if all the eigenvalues of the matrix

$$\begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ I & 0 \end{pmatrix} \tag{51}$$

are inside the unit circle. For the period $1960 - 79$, the $\tilde{b}_1$ and $\tilde{b}_2$ matrices are

$$\tilde{b}_1 = \begin{pmatrix} .41 & .50 & -1.5 \\ .44 & .54 & -.82 \\ .12 & .15 & .45 \end{pmatrix}; \tilde{b}_2 = \begin{pmatrix} 0 & 0 & 1.02 \\ 0 & 0 & 1.10 \\ 0 & 0 & .30 \end{pmatrix}.$$

The maximum eigenvalue of (51) is .95 so that this solution is stationary. However, the eigenvalues of (43) have a pair of complex conjugates with real parts 2.1 so that the solution is not E-stable. This shows that even though there exist stationary sunspot equilibria in the pre-Volcker period, they are not learnable by private agents.

It can also be shown that even the fundamental equilibria are not E-stable for these parameter configurations. The MSV (fundamental) solutions take the form

$$x_t = bx_{t-1} + cw_t$$

and solving the matrix quadratic, $\Omega b^2 - b + \delta = 0$; yields two stationary MSV solutions for b given by

$$\begin{pmatrix} 0 & 0 & -1.5 \\ 0 & 0 & -.82 \\ 0 & 0 & .45 \end{pmatrix} \text{and} \begin{pmatrix} 0 & 0 & .19 \\ 0 & 0 & 1.01 \\ 0 & 0 & .95 \end{pmatrix}. \tag{52}$$

If agents have a PLM of the form

$$x_t = a + bx_{t-1} + cw_t \tag{53}$$

then a necessary condition for E-stability is that the eigenvalues of the matrix $- + -\bar{b}^{1}$ have real parts less than one if agents use last period data on output, inflation, and interest rates to form their forecasts. However, it is easy to check that this condition is violated for both the solutions given in (52). We note here that the estimated rule in the pre-Volcker era fails the Taylor principle and we (again) find all types of RE solutions, both MSV and SSEs, unstable under learning.[23]

These results offer a novel explanation for the high inflation in the pre-Volcker era. Since neither the fundamental nor the non-fundamental equilibria are E-stable, the high inflation of the 1960s and 1970s may have been due to the persistent learning dynamics of private sector agents. The forecasting errors made by agents did not disappear over time owing to the monetary policy being pursued by the Federal Reserve. On this interpretation, these errors were not due to the economy being in a sunspot equilibrium, as suggested in (Clarida, Gali, and Gertler 2000).

In contrast, in the Volcker-Greenspan era the monetary policy followed was not compatible with the existence of a stationary sunspot REE. Using the baseline estimates of $\chi_{\pi} = 2.15$; $\chi_{z} = .93$ and $\theta = .79$ in Table 2 of (Clarida, Gali, and Gertler 2000) for this period, one can check that there exists only one stationary MSV solution, namely

$$\begin{pmatrix} 0 & 0 & -1.22 \\ 0 & 0 & -0.64 \\ 0 & 0 & 0.43 \end{pmatrix};$$

which is E-stable if agents have a PLM of the form (53). This monetary policy satisfies the Taylor principle and was conducive to learnability of the unique MSV solution, which may in fact have contributed to the low inflation during this period.

6 Discussion and Concluding Remarks

We have carried out a general analysis of learnability of non-fundamental equilibria for multivariate forward-looking linear models with and without lags. Our results apply to models of monetary policy that are being used to give advice to policy makers. Learnability of the fundamental REE and unlearnability of non-fundamental REE are an important constraint that good monetary policy design should aim to meet. Otherwise undesirable fluctuations may result.

While it is clearly desirable to achieve learnability of the MSV REE by appropriate choice of the policy rule, the assessment of the fluctuations arising as non-fundamental REE is less clear-cut. While these endogenous fluctuations are usually inferior to MSV REE, their normative comparison is ambiguous with respect to cases where the economy fluctuates as a result of there being no learnable REE. Nevertheless, it would seem possible to conduct a positive analysis between stable SSEs and non-rational fluctuations due to non-learnable equilibria.

[23] This rule is inertial but it fails the Taylor principle; see Chapter 4 of (Woodford 2003b) for the definition of the Taylor principle in this case. Similar conclusions follow if agents use contemporaneous information on output and inflation to form their forecasts.

The scenario in (Clarida, Gali, and Gertler 2000) illustrates the possibilities for a positive analysis. Clarida et al suggest that the high and volatile U.S. in°ation in the 1960s/70s may have been indeterminate equilibria caused by the policy. Our analysis has shown that neither the fundamental nor the non-fundamental REE were learnable during this period. The volatile period was perhaps a situation of agents trying unsuccessfully to ¯nd some equilibrium and not necessarily a sunspot equilibrium.

A further analysis of this issue would seem worth while. The (Clarida, Gali, and Gertler 2000) explanation is based on RE and it would be consistent with agents not making systematic errors in their forecasts of in°ation and output gap. Our explanation has agents making forecast errors that do not disappear over time. Agents might believe in PLMs corresponding to the fundamental REE but, since errors do not disappear over time, they might also entertain the possibility of PLM matching the form of some non-fundamental REE. However, even in the latter case, the forecast errors would not disappear over time as all REE are unstable under learning. Thus, one way to test the competing hypotheses is to study the behavior of forecast errors in in°ation and output gaps in the pre-Volcker era.

We have also found policy rules and domains for policy parameters which satisfy the Taylor principle but are nevertheless associated with indeterminacy and existence of learnable non-fundamental REE. Both fundamental and some non-fundamental REE, are potentially learnable for some domains of policy parameters under the rules considered in Sections 3.1 and 3.2. This result shows the Taylor principle does not always guarantee determinacy and it is important to avoid indeterminacies when forward-looking policy rules conform to the Taylor principle. Indeterminacy can be avoided with moderate aggression to in°ation and/or output gap forecasts. Furthermore, even when indeterminacies exist, a modest response to output leads to instability of all types of sunspots (as in Section 3.1) and to learnability of the MSV solution.

A further way to reduce indeterminacy is to make the interest rule react directly to its own past values which makes it easier to satisfy the Taylor principle. These inertial rules have been found to have desirable properties: they can lead to the existence of a unique learnable fundamental equilibrium and also have the potential to implement optimal policy of the central bank; see (Bullard and Mitra 2001) and (Rotemberg and Woodford 1999). The U.S. interest rule estimated since the 1980s by (Clarida, Gali, and Gertler 2000) has such an inertial component that, in conjunction with an appropriate response to the in°ation forecast and output gap, leads to a unique learnable fundamental REE.

In summary, we do not advocate policies that violate the Taylor principle. Policies satisfying the Taylor principle are recommended as long as they do not lead to indeterminacy. In addition, our results in Section 3.2 suggest that in°ation-targeting central banks should adopt a policy of °exible in°ation targeting instead of strict in°ation targeting since the latter can lead to the existence of learnable, indeterminate equilibria. This is what most in°ation-targeting central banks seem to do in practice.

A Appendices: Derivations

A.1 Proof of Proposition 1

Define the new variables $p_t = Q^{-1}y_t$.[24] This allows us to write the system (4) in the form

$$p_t = \Lambda p_{t-1} + Q^{-1}u_t. \tag{54}$$

The second equation of (54) can then be written as

$$p_{2,t} = \lambda_2 p_{2,t-1} + Q^{21}\eta_{1,t} + Q^{22}\eta_{2,t} + Q^{23}v_{1,t} + Q^{24}v_{2,t},$$

where the notation $Q^{-1} = (Q^{ij})$ has been used. Stationarity implies the restriction $p_{2,t} = 0$ or

$$Q^{21}x_{1,t} + Q^{22}x_{2,t} + Q^{23}w_{1,t} + Q^{24}w_{2,t} = 0. \tag{55}$$

The first equation is

$$p_{1,t} = \lambda_1 p_{1,t-1} + Q^{11}\eta_{1,t} + Q^{12}\eta_{2,t} + Q^{13}v_{1,t} + Q^{14}v_{2,t}. \tag{56}$$

These imply that one of components of the martingale difference sequence η_t is a linear combination of the other component and the iid shocks to the exogenous variables. Using the definition

$$p_{1,t} = Q^{11}x_{1,t} + Q^{12}x_{2,t} + Q^{13}w_{1,t} + Q^{14}w_{2,t}$$

we can write (56) as

$$\begin{aligned} Q^{11}x_{1,t} + Q^{12}x_{2,t} = \; & \lambda_1 Q^{11}x_{1,t-1} + \lambda_1 Q^{12}x_{2,t-1} - Q^{13}w_{1,t} - Q^{14}w_{2,t} \\ & + \lambda_1 Q^{13}w_{1,t-1} + \lambda_1 Q^{14}w_{2,t-1} + \\ & Q^{11}\eta_{1,t} + Q^{12}\eta_{2,t} + Q^{13}v_{1,t} + Q^{14}v_{2,t}. \end{aligned}$$

This equation and (55) make up the system (5) in the text.

A.2 Relation Between Markov and Autoregressive REE

First, we note that solutions to linear model (1)-(2) can be thought as the sum of a particular solution and a general solution to the homogenous equation. We take the MSV solution

$$x_t^M = \bar{f} w_t, \text{ where } \bar{f} = \bar{f}\Psi + \Phi;$$

[24] This is a modification of the well-known Blanchard-Kahn technique for obtaining stationary solutions to regular (i.e. "saddle-point stable") multivariate linear RE models. See, Appendix 2 of Chapter 10 in (Evans and Honkapohja 2001) for the extension of the technique to irregular models.

as the particular solution. The general solution to the homogenous equation is any process x_t^H satisfying $x_t^H = \beta E_t x_{t+1}^H$. Introducing an arbitrary innovation, we write

$$x_t^H = \beta^{-1} x_{t-1}^H + \varepsilon_t, \qquad (57)$$

where ε_t is a MDS. Let $x_t^G = x_t^M + x_t^H$. Then

$$x_t^G = \beta^{-1} x_{t-1}^G + \varepsilon_t + f w_t - \beta^{-1} f w_{t-1}.$$

Using the equation $f = \beta f \Psi + \Phi$ and (2), the last terms can be written as

$$\varepsilon_t + f w_t - \beta^{-1} f w_{t-1} = \varepsilon_t + f v_t - \beta^{-1} \Phi w_{t-1}.$$

It follows that x_t^G has the same form as (3), when we set $\eta_t = \varepsilon_t + f v_t$. η_t is a MDS since ε_t is a MDS and v_t is iid.

Second, the relationship between the autoregressive and Markov solutions is obtained by choosing the finite-state Markov process as a specific solution to the homogenous equation. We set $x_t^H = x^H(i)$ if $s_t = i$ where $x^H(i) = \beta \sum_{i=1}^{K} \Pi_{ij} x^H(j)$ and use the well-known result that Markov chains can be written in autoregressive form (57) with an MDS innovation.

A.3 Mathematica Routine used in Proposition 4

We give a brief description of the Mathematica routine (available on request) used in computing the eigenvalues of $DT_b(\bar{b})$. For computing $\bar{b}$, we need only the top left 2×2 block of the diagonalization matrix for B, namely Q. In addition, since B is block triangular, this matrix corresponds to the diagonalization of β^{-1}. Denote the 2×2 matrix $\beta = (\beta_{ij})$. The Jordan decomposition on β^{-1} yields the following diagonalization matrix

$$M = \begin{pmatrix} \frac{\beta_{11} - \beta_{22} + \sqrt{\beta_{11}^2 + \beta_{22}^2 - 2\beta_{11}\beta_{22} + 4\beta_{12}\beta_{21}}}{2\beta_{21}} & \frac{\beta_{11} - \beta_{22} - \sqrt{\beta_{11}^2 + \beta_{22}^2 - 2\beta_{11}\beta_{22} + 4\beta_{12}\beta_{21}}}{2\beta_{21}} \\ 1 & 1 \end{pmatrix}.$$

Note that, as mentioned above, M coincides with the top left 2×2 block of Q. The eigenvalues of β^{-1} are

$$\lambda_1 = \frac{\beta_{11} + \beta_{22} - \sqrt{\beta_{11}^2 + \beta_{22}^2 - 2\beta_{11}\beta_{22} + 4\beta_{12}\beta_{21}}}{2(\beta_{11}\beta_{22} - \beta_{12}\beta_{21})},$$

$$\lambda_2 = \frac{\beta_{11} + \beta_{22} + \sqrt{\beta_{11}^2 + \beta_{22}^2 - 2\beta_{11}\beta_{22} + 4\beta_{12}\beta_{21}}}{2(\beta_{11}\beta_{22} - \beta_{12}\beta_{21})}.$$

In general, we do not know whether λ_1 or λ_2 has the smaller modulus. Assume for now $|\lambda_1| < 1$, $|\lambda_2| > 1$. With this,

$$\bar{b} = M \begin{pmatrix} \lambda_1 M_{11}^{-1} & \lambda_1 M_{12}^{-1} \\ 0 & 0 \end{pmatrix}$$

where M_{ij}^{-1} denotes the (i, j) element of M^{-1} and $\bar{b}$ coincides with (21). It is then easy to check that one of the eigenvalues of $DT_b(\bar{b}) = \bar{b}' \otimes \beta + I \otimes \beta \bar{b}$ is 2.

A.4 Mathematical Lemma used in Proposition 6

Consider the following linear system of differential equations

$$\dot{x} = Ax \tag{58}$$

with x an n dimensional vector. We assume that A can be written in the form $A = Q\Lambda Q^{-1}$, where the matrix of eigenvalues takes the form

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{59}$$

and all $n-1$ eigenvalues of Λ_1 have negative real parts. Λ_1 is thus invertible. Partition $Q^{-1} = (Q^{ij})$ as

$$Q^{-1} = \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}$$

where Q^{11} is $(n-1) \times (n-1)$, Q^{12} is $(n-1) \times 1$, Q^{21} is $1 \times (n-1)$ and Q^{22} is a non-zero scalar. We assume that the matrix $Q^{11} - Q^{12}(Q^{22})^{-1}Q^{21}$ is invertible.

We state the following auxiliary result:

Lemma 8 For any initial condition $x(0)$ the trajectory $x(t \mid x(0))$ of (58) converges to the set of equilibrium points $\{\hat{x} \mid A\hat{x} = 0\}$.

We have not discovered this result in the mathematics literature, though we suspect that it is a known result. A proof is available on request.

A.5 Details on Section 5

For the computation of irregular equilibria in the model of (Clarida, Gali, and Gertler 2000), we follow the technique illustrated in Chapter 10, Appendix 2 of (Evans and Honkapohja 2001). We factor J as $\Lambda = Q^{-1}JQ$; where $Q^{-1} = \{q^{ij};\ i,j = 1,\ldots,6\}$ and Λ are correspondingly partitioned as

$$Q^{-1} = \begin{pmatrix} Q^{11}(1,1) & Q^{11}(1,2) & Q^{12}(1) \\ Q^{11}(2,1) & Q^{11}(2,2) & Q^{12}(2) \\ Q^{21}(1) & Q^{21}(2) & Q^{22} \end{pmatrix}; \quad \Lambda = \begin{pmatrix} \Lambda_1^{*} & 0 & 0 \\ 0 & \Lambda_1^{\#} & 0 \\ 0 & 0 & \Lambda_2 \end{pmatrix}.$$

Note that the diagonal matrix Λ_1^{*} above contains the eigenvalues of J with modulus less than one whereas $\Lambda_1^{\#}$ and Λ_2 are diagonal matrices containing the eigenvalues of J with modulus more than one. The free variables are also partitioned into the sets

$$x_t^1 = \begin{pmatrix} x_t^{1*} \\ x_t^{1\#} \end{pmatrix}.$$

If we use the baseline estimates in Table 2 of (Clarida, Gali, and Gertler 2000) for the period 1960 − 79, the eigenvalues of J happen to be $\lambda_1 = 0$, $\lambda_2 = .63$, $\lambda_3 = 1.05$, $\lambda_4 = 2.21$, $\lambda_5 = \gamma^{-1}$, $\lambda_6 = \rho^{-1}$; i.e., exactly 2 eigenvalues of J are inside the unit circle. Assume that $\Lambda_1^* = \{\lambda_1, \lambda_2\}$; $\Lambda_1^\# = \{\lambda_3\}$; and $\Lambda_2 = \{\lambda_4, \lambda_5, \lambda_6\}$. Here we have $x_t^{1*} = \{z_t, \pi_t\}$; $x_t^{1\#} = \{i_t\}$. It can be checked that the final solution for $x_t = (z_t, \pi_t, i_t)'$ may be written as (omitting the shocks)

$$x_t = \Phi_1 x_{t-1} + \Phi_2 x_{t-2} + \ldots$$

which is a vector ARMA process. Introducing the notation $(Q^{11})^{-1} = \{q_{ij};\ i, j = 1, \ldots, 3\}$, we have

$$\Phi_1 = \begin{pmatrix} \lambda_3^{-1} q_{13} q^{31} & \lambda_3^{-1} q_{13} q^{32} & q_{13}(\lambda_3^{-1} q^{33} - q^{34}) - q_{11} q^{14} - q_{12} q^{24} \\ \lambda_3^{-1} q_{23} q^{31} & \lambda_3^{-1} q_{23} q^{32} & q_{23}(\lambda_3^{-1} q^{33} - q^{34}) - q_{21} q^{14} - q_{22} q^{24} \\ \lambda_3^{-1} q_{33} q^{31} & \lambda_3^{-1} q_{33} q^{32} & q_{33}(\lambda_3^{-1} q^{33} - q^{34}) - q_{31} q^{14} - q_{32} q^{24} \end{pmatrix} \quad (60)$$

$$\Phi_2 = \begin{pmatrix} 0 & 0 & \lambda_3^{-1} q_{13} q^{34} \\ 0 & 0 & \lambda_3^{-1} q_{23} q^{34} \\ 0 & 0 & \lambda_3^{-1} q_{33} q^{34} \end{pmatrix}. \quad (61)$$

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