Outline:

Today’s Class

- Linear Equations in Two Unknowns:
- Non-Linear Equations
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● Linear Equations in Two Unknowns:

● Non-Linear Equations
We now review some methods for solving two linear equations with two unknowns.

For example, find the values of $x$ and $y$ that satisfy both equations.

\begin{align*}
2x + 3y &= 18 \\
3x - 4y &= -7
\end{align*}
Method 1

Solve one of the equations for one of the variables in terms of the other; then substitute the result into the other equation.

This leaves only one equation in one unknown which is easily solvable.

Let's apply this method to our system by solving the first equation for \( y \) in terms of \( x \).
We get \( y = 6 - \frac{2}{3}x \).

Substituting this into the second equation gives us:

\[
3x - 4\left(6 - \frac{2}{3}x\right) = -7
\]

\[
3x - 24 + \frac{8}{3}x = -7
\]

\[
9x - 72 + 8x = -21
\]

Finally:

\[
17x = 51 \iff x = 3
\]
Method 2

This method is based on eliminating one of the variables by adding or subtracting a multiple of one equation from the other.

Suppose we want to eliminate $y$ from our system.

If we multiply the first equation by 4 and the second by 3, then the coefficients will be the same except from the sign.

If we add the transformed equations, the term in $y$ disappears.
Linear Equations in Two Unknowns:

Introduction:

We obtain:

\[ 8x + 12y = 72 \]
\[ 9x - 12y = -21 \]
\[ 17x = 51 \]

Hence, \( x = 3 \) (as before).

To find the value for \( y \), substitute 3 for \( x \) in either of the original equations and solve for \( y \).
We can now use method 2 to solve a general linear system with two equations and two unknowns:

\[ ax + by = c \]
\[ dx + ey = f \]

Here, \( a, b, c, d, e \) and \( f \) are arbitrary given numbers, whereas \( x \) and \( y \) are the unknowns.
We multiply the first equation by $e$ and the second by $-b$ to obtain:

\[
\begin{align*}
  aex + bey &= ce \\
  -bdx - bey &= -bf \\
  \hline
  (ae - bd)x &= ce - bf
\end{align*}
\]

Which gives the value for $x$. 

We can substitute back into the first equation to find \( y \), the result is:

\[
\begin{align*}
  x &= \frac{ce - bf}{ae - bd}, \\
  y &= \frac{af - cd}{ae - bd}
\end{align*}
\]

We have found expressions for both \( x \) and \( y \).
Linear Equations in Two Unknowns:

Introduction:

PROBLEMS FOR SECTION 2.4

Solve the systems of equations in 1–3:

1. (a) \[ x - y = 5 \]
   \[ x + y = 11 \]
   (b) \[ 4x - 3y = 1 \]
   \[ 2x + 9y = 4 \]
   (c) \[ 3x + 4y = 2.1 \]
   \[ 5x - 6y = 7.3 \]

2. (a) \[ 5x + 2y = 3 \]
   \[ 2x + 3y = -1 \]
   (b) \[ x - 3y = -25 \]
   \[ 4x + 5y = 19 \]
   (c) \[ 2x + 3y = 3 \]
   \[ 6x + 6y = -1 \]
Outline:

Today’s Class

- Linear Equations in Two Unknowns:
- Non-Linear Equations
Non-Linear Equations:

Introduction:

We now consider some additional types of equation that you will encounter later, particularly in connection with optimization problems.

**Example 1**

Solve each of the following three separate equations:

(a) \( x^3 \sqrt{x + 2} = 0 \)  
(b) \( x(y + 3)(z^2 + 1)\sqrt{w - 3} = 0 \)  
(c) \( x^2 - 3x^3 = 0 \)

**Solution:**

(a) If \( x^3 \sqrt{x + 2} = 0 \), then either \( x^3 = 0 \) or \( \sqrt{x + 2} = 0 \). The equation \( x^3 = 0 \) has only the solution \( x = 0 \), while \( \sqrt{x + 2} = 0 \) gives \( x = -2 \). The solutions of the equation are therefore \( x = 0 \) and \( x = -2 \).

(b) There are four factors in the product. One of the factors, \( z^2 + 1 \), is never 0. Hence, the solutions are: \( x = 0 \) or \( y = -3 \) or \( w = 3 \).

(c) Start by factoring: \( x^2 - 3x^3 = x^2(1 - 3x) \). The product \( x^2(1 - 3x) \) is 0 if and only if \( x^2 = 0 \) or \( 1 - 3x = 0 \). Hence, the solutions are \( x = 0 \) and \( x = 1/3 \).
Non-Linear Equations:

Introduction:

In general:

\[ ab = ac \]

is equivalent to \[ a = 0 \] or \[ b = c \]

because the equation \[ ab = ac \] is equivalent to \[ ab - ac = 0 \], or \[ a(b - c) = 0. \]

This product is 0 when \[ a = 0 \] or \[ b = c \].

If \[ ab = ac \] and \[ a \neq 0 \], we conclude that \[ b = c \].
What conclusions about the variables can we draw if

(a) \( x(x + a) = x(2x + b) \)
(b) \( \lambda y = \lambda z^2 \)
(c) \( xy^2(1 - y) - 2\lambda(y - 1) = 0 \)

Solution:

(a) \( x = 0 \) or \( x + a = 2x + b \). The last equation gives \( x = a - b \). The solutions are therefore \( x = 0 \) and \( x = a - b \).
(b) \( \lambda = 0 \) or \( y = z^2 \). (It is easy to “forget” the possibility that \( \lambda = 0 \).)
(c) The equation is equivalent to

\[
xy^2(1 - y) + 2\lambda(1 - y) = 0, \quad \text{that is} \quad (1 - y)(xy^2 + 2\lambda) = 0
\]

We conclude from the last equation that \( 1 - y = 0 \) or \( xy^2 + 2\lambda = 0 \), that is \( y = 1 \) or \( \lambda = -\frac{1}{2} xy^2 \).
Non-Linear Equations:

Introduction:

**EXAMPLE 3**  Solve the following equations:

(a) \[ \frac{1 - K^2}{\sqrt{1 + K^2}} = 0 \]

(b) \[ \frac{45 + 6r - 3r^2}{(r^4 + 2)^{3/2}} = 0 \]

(c) \[ \frac{x^2 - 5x}{\sqrt{x^2 - 25}} = 0 \]

**Solution:**

(a) The denominator is never 0, so the fraction is 0 when \(1 - K^2 = 0\), that is when \(K = \pm 1\).

(b) Again the denominator is never 0. The fraction is 0 when \(45 + 6r - 3r^2 = 0\), that is \(3r^2 - 6r - 45 = 0\). Solving this quadratic equation, we find that \(r = -3\) or \(r = 5\).

(c) The numerator is equal to \(x(x - 5)\), which is 0 if \(x = 0\) or \(x = 5\). At \(x = 0\) the denominator is \(\sqrt{-25}\), which is not defined, and at \(x = 5\) the denominator is 0. We conclude that the equation has no solutions.