Note Title 6/17/2013

Some Aspects of the Kibble-Zurek Problem

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The Kibble-Zurek Problem: Universality and the Scaling Limit

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Kibble-Zurek Scaling and String-Net Coarsening in Topologically Ordered Systems

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(Dated: April 9, 2013)

A. What is the KZ problem?

B. The KZ scaling limit

C. KZ with symmetry breaking

D. KZ without symmetry breaking

What is the KZ problem?

Kibble
Zurek

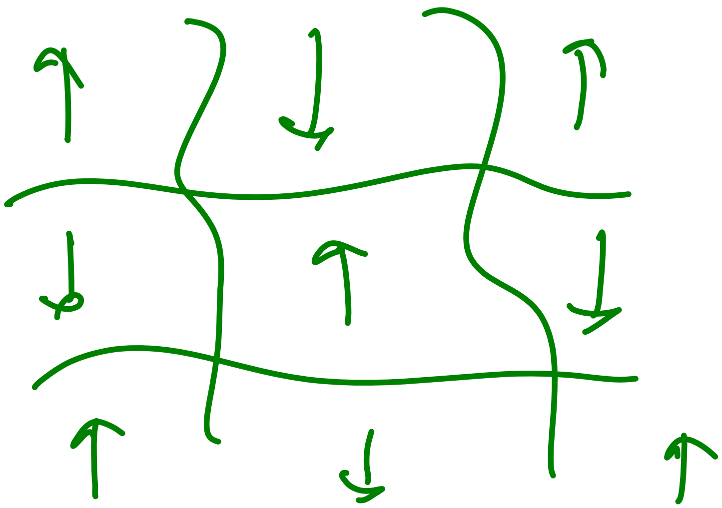
KZ mechanism

symmetric | broken symmetry

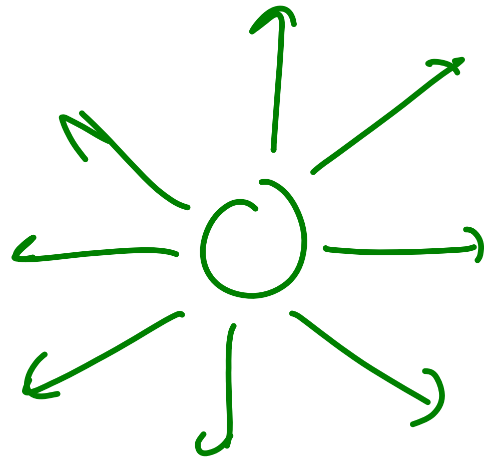
T_c

$\longrightarrow$ time (expansion)

Fall out of equilibrium near T_c, arrive at $T < T_c$ with finite sized domain with different local orientations of broken symmetry



$\mathbb{Z}_2$



$U(1)$

and defects to accomodate the mismatch.

Generalized to quantum phase transitions by Polkovnikov, Zurek, Dziarmaga, Damski

Typically discussed as

$$\text{density of defects} \propto \left(\frac{\text{control parameter}}{\text{velocity}}\right)^{\sigma}$$

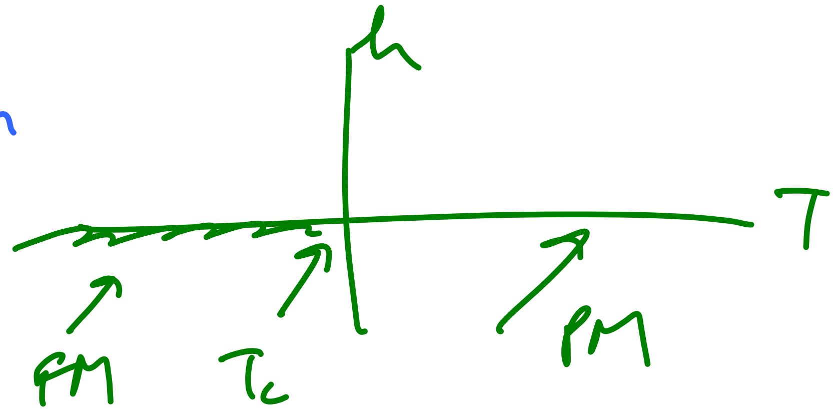
universal exponent

Cosmic strings, CM analogs

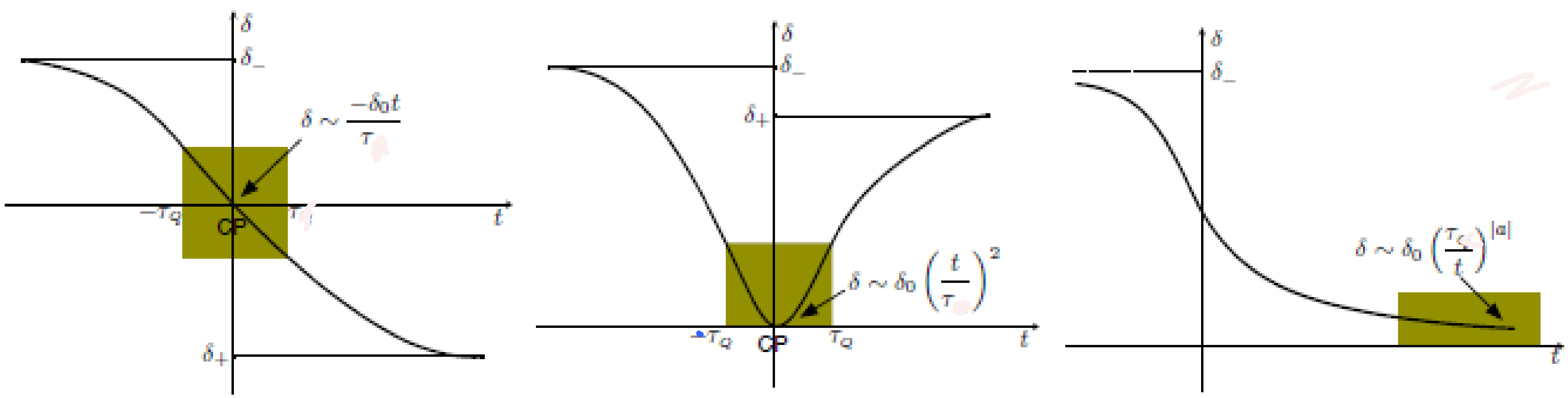
<u>KZ problem</u> Start in equilibrium/ground state near C.P. Vary all possible $\delta_i(t)$ passing asymptotically close to the C.P. Calculate resulting non-equilibrium states, "thermodynamic" variables and correlation functions

Sharpen: Protocol + Scaling limit

E.g. FM transition



$\delta = h$ or $T - T_c$



Trans-critical protocol Cis-critical End-critical

The KZ Scaling Limit

Work near CP where universal physics is expected to emerge. Go slowly ($\tau \to \infty$) so that equilibrium is lost asymptotically close to the CP.

Bare time scale τ gives rise to an emergent time scale t_{KZ} and length l_{KZ}

At δ, relaxation time $\xi_t(\delta) = \xi^z(\delta) = \delta^{-\nu z}$

When $\delta(t)$ changes need $\Delta\xi_t$ over time $\xi_t \ll \xi_t$ to remain adiabatic.

$$\Rightarrow \quad \frac{d\xi_t}{dt} \cdot \xi_t \ll \xi_t \quad \text{or} \quad \frac{d\xi_t}{dt} \ll 1$$

Adiabaticity lost when $t = t_{KZ} = \tau^{\nu z / \nu z + 1}$ linear TCP

Correlation length at $t = t_{KZ}$ is $\ell_{KZ} = t_{KZ}^{1/z}$

KZ time and length

$$\delta(t) = \delta_0 \left(-\frac{t}{\tau}\right)^a$$

$$\Rightarrow \xi_t(t) = \delta^{-\nu z} = \left[\delta_0\left(-\frac{t}{\tau}\right)^a\right]^{-\nu z}$$

$$\Rightarrow \dot{\xi}_t = -\frac{1}{t}\,\xi_t = 1 \quad \text{when } \xi_t = |t| \quad (t<0)$$

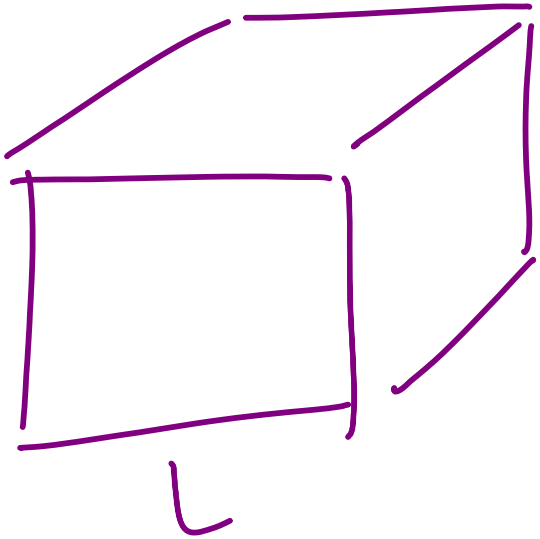
Solve $\left[\delta_0\left(\frac{|t|}{\tau}\right)^a\right]^{-\nu z} = |t|$ to get

$$t_{kz} = \left(\frac{\tau}{\delta_0^{1/a}} \right)^{\frac{a\nu z}{1+a\nu z}}$$

$$\ell_{kz} = \xi(t_{kz}) = \left[\xi_t(t_{kz}) \right]^{1/z} = t_{kz}^{1/z}$$

Note that $t_{kz} \ll \tau$ as $\tau \to \infty$.

Recall finite size scaling



$$\langle O \rangle = \frac{1}{L^{\Delta}} f(\delta L^{1/\nu})$$

$L/3$

Generalize.

KZ scaling limit

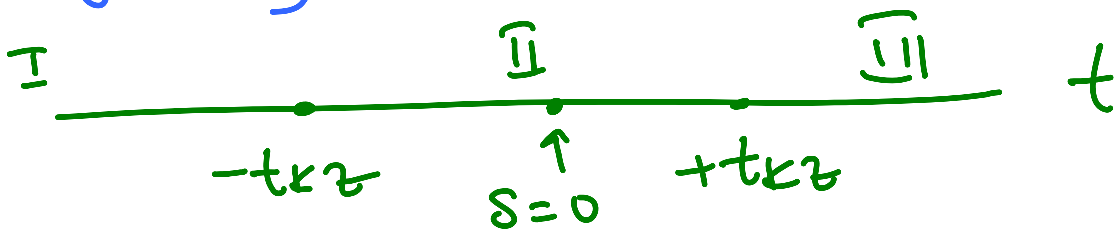
$\tau \to \infty$, $t_{KZ} \to \infty$, $l_{KZ} \to \infty$ keeping $\frac{x}{l_{KZ}}$, $\frac{t}{t_{KZ}}$ fixed

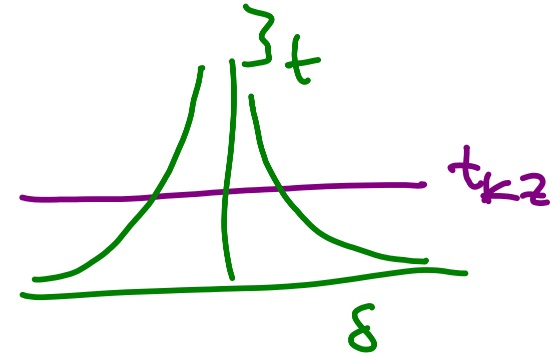
$$\Rightarrow \langle O(\bar{x}t) \rangle_\tau \sim \frac{1}{l_{KZ}^{\Delta}} \tilde{G}_0\left(\frac{t}{t_{KZ}}\right)$$

$$\langle O(\bar{x}t)\, O(\bar{x}'t') \rangle \sim \frac{1}{l_{KZ}^{2\Delta}} \tilde{G}_0\left(\frac{|\bar{x}-\bar{x}'|}{l_{KZ}}, \frac{t}{t_{KZ}}, \frac{t'}{t_{KZ}}\right)$$

$$\xi_{ne}(t, c) \sim l_{KZ}\, \mathcal{L}\left(\frac{t}{t_{KZ}}\right) \qquad \text{etc}$$

Scaling functions have information on entire trajectory





For $\frac{t}{t_{KZ}} \ll 1$ match equilibrium I

$\frac{t}{t_{KZ}} \gg 1$ "back to equilibrium" III

$\frac{t}{t_{KZ}} \sim 1$ "KZ critical" II

Conjecture

Universality class : Equilibrium U C + protocol

Aim of theory

Establish universality and compute all scaling functions of interest

C. KZP with symmetry breaking

Back to FM

Consider classical Model A dynamics + TCP

$$F[\phi] = \int d^dx \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} r \phi^2 + u\phi^4 \right]$$

$$\frac{\partial \phi(x,t)}{\partial t} = -\frac{\delta F}{\delta \phi(x,t)} + \zeta(x,t)$$

$$\langle \zeta(\bar{x},t)\, \zeta(\bar{x}',t') \rangle = \delta^{(d)}(\bar{x}-\bar{x}')\, \delta(t-t')$$

with $r = -\tanh\left(\frac{t}{\tau}\right)$ for specificity.

Gaussian theory $u=0$ $\nu=\frac{1}{2}$, $z=2$

Solve linear PDE

$$\phi(\vec{k}\,t) = \int_{-\infty}^{t} dt' \, e^{-\int_{t'}^{t} dt''\left[k^2+r(t'')\right]} \zeta(\vec{k}\,t')$$

Then noise averaged correlator

$$G_{\phi\phi}(k,t;\tau) = 2\int_{-\infty}^{t} dt' \, e^{-2\int_{t'}^{t} dt''\left[k^2+r(t'')\right]}$$

τ

Now $-\tanh(t/\tau) \sim -(t/\tau)$ as long as $|t| \ll \tau$

and still in equilibrium/adiabatic " $|t| \gg t_{KZ}$

Since $t_{KZ} = \tau^{1/2} \ll \tau$ the system is <u>well</u> inside the linear region of the protocol before any departures from equilibrium are visible.

$\Rightarrow$ replace $-\tanh\left(\frac{t}{\tau}\right)$ by $-\frac{t}{\tau}$ WLOG as $\tau \to \infty$.

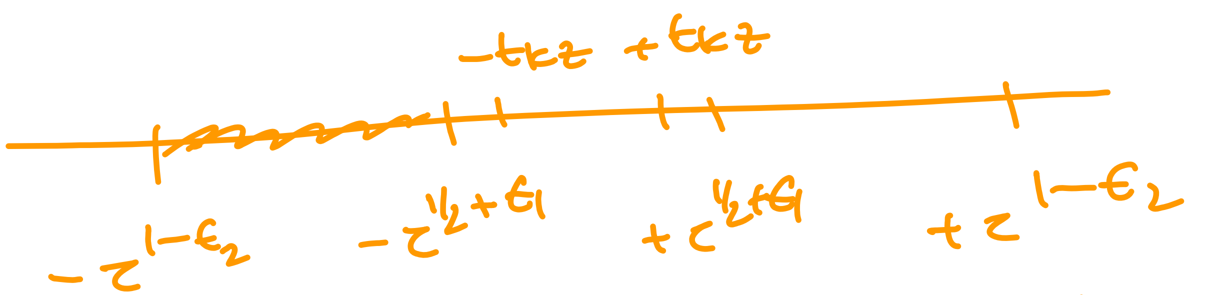
Since this argument can be made for <u>any</u> smooth $\gamma(t)$ with $\gamma'(0) \neq 0$ universality wrt choice of asymptotically linear protocol follows

Formally $G^{ad}_{tt} = \dfrac{1}{k^2 + r(t)}$ for $-t \gg \tau^{1/2+\epsilon_1}$ (say)

can be matched to the exact solution of

$r(t) = -t/\tau$ valid for $-t \ll \tau^{1-\epsilon_2}$

over $\tau^{1-\epsilon_2} \gg |t| \gg \tau^{1/2-\epsilon_1}$



to construct a global solution valid on

$(-\infty, +\tau^{1-\epsilon_2})$

Back to $G_{\phi\phi}$

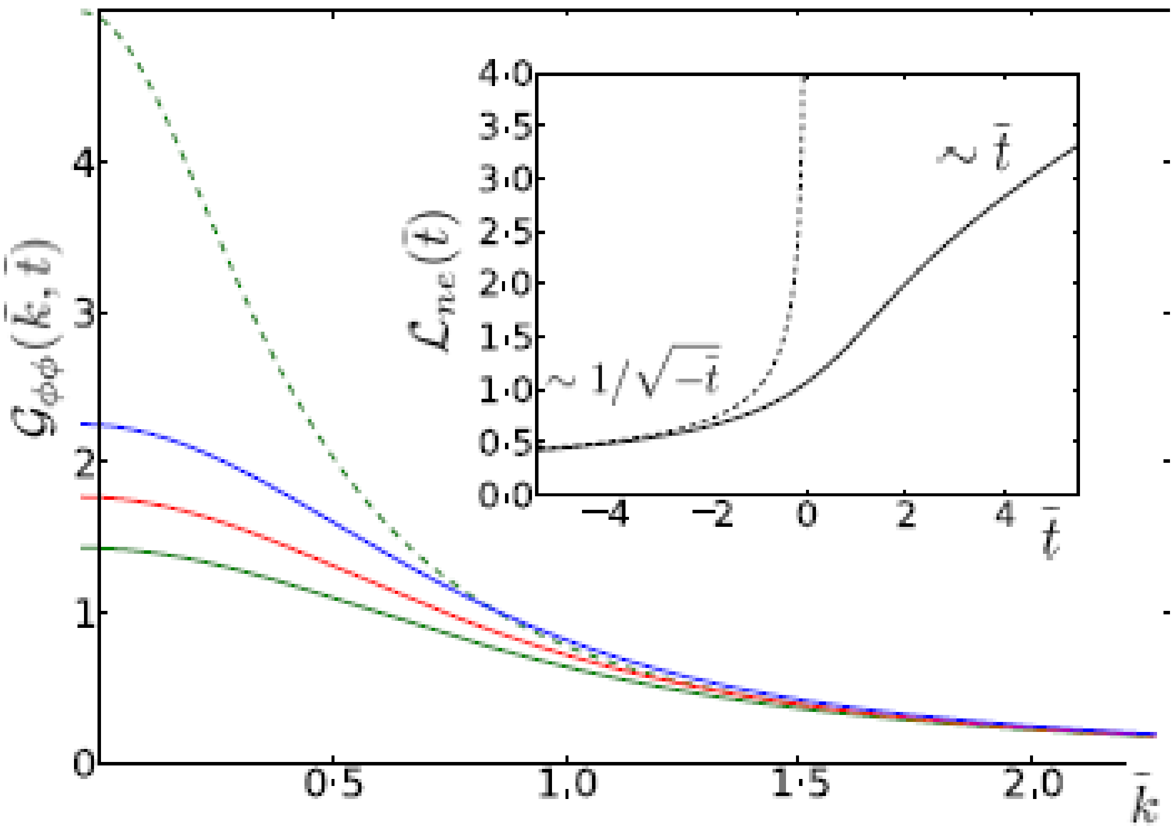
$$G_{\phi\phi}(k,t;\tau) = 2\int_{-\infty}^{t} dt' \, e^{-2\int_{t'}^{t} dt''\left[k^2 - \frac{t''}{\tau}\right]}$$

Change variables $z' = t'/t_{\kappa z}$, $z'' = t''/t_{\kappa z}$

$$G_{\phi\phi}(k,t;\tau) = 2 t_k \int_{-\infty}^{t/t_k} dz' \, e^{-2\int_{z'}^{z''} dz''\left(k l_{\kappa z}^2 - z''\right)}$$

$$= t_k \, \tilde{G}_{\phi\phi}\left(k l_{\kappa z}^2, t/t_{\kappa z}\right)$$

where $l_{\kappa z} = t_{\kappa z}^{1/2} = \tau^{1/4}$.



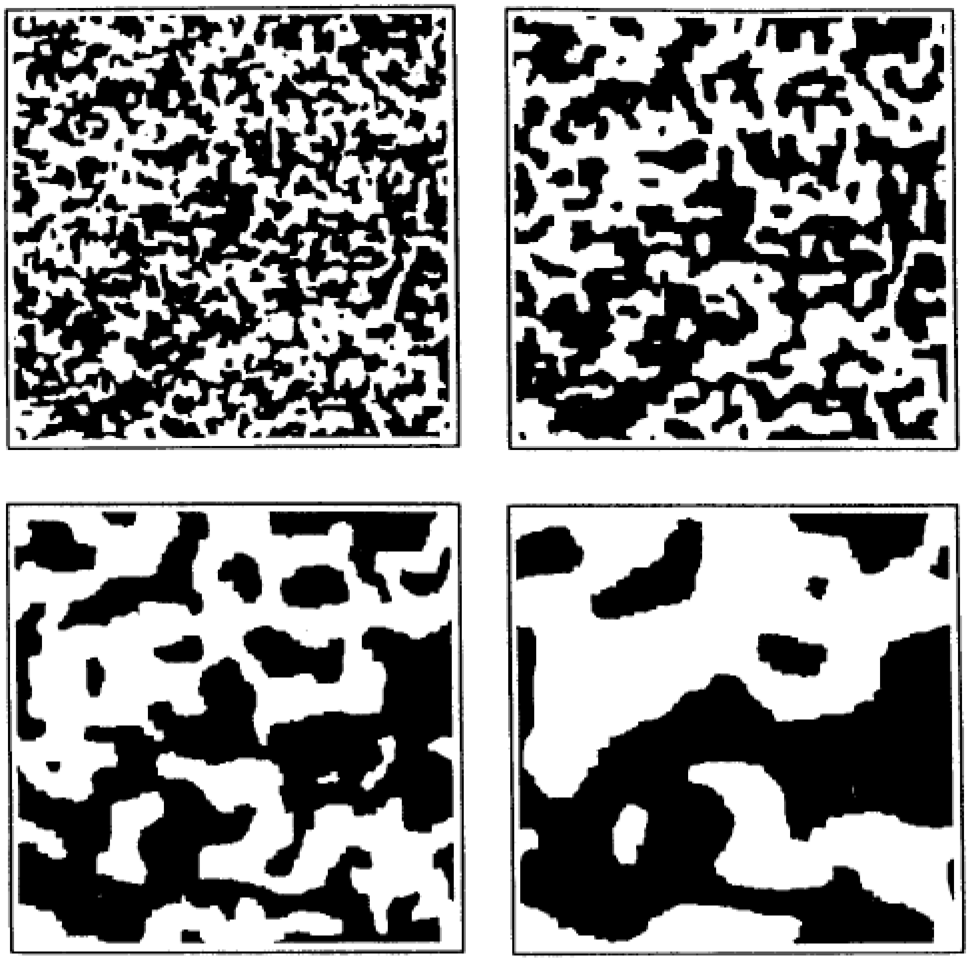
$G_{\phi\phi}$ is finite for $r(t) < 0$!

OK for KZ scaling regime for $d \geq 4$

but for $d < 4$ need quartic term

and the physics of coarsening in

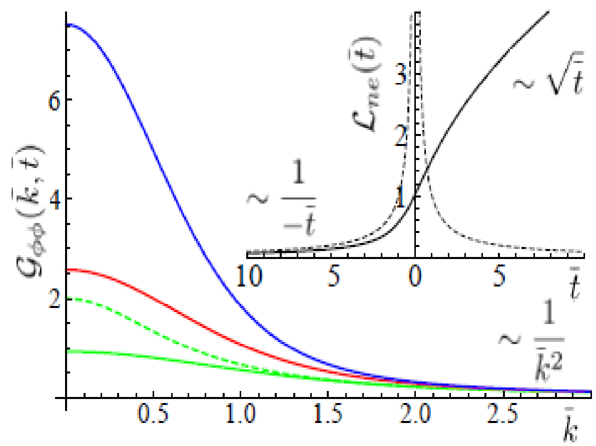
region $\underline{\text{III}}$ (back to equilibrium)



$$l_{co}(t) \sim \xi \left(\frac{t}{\xi_t}\right)^{1/z_d}$$

Figure 2. Monte Carlo simulation of domain growth in the $d = 2$ Ising model at $T = 0$ (taken from Kissner [8]). The system size is 256×256, and the snapshots correspond to 5, 15, 60 and 200 Monte Carlo steps per spin after a quench from $T = \infty$.

To this end we have studied the $O(N \to \infty)$ limit where $\nu = \frac{1}{d-2}$, $z=2$, $z_d=2$ and $G_{\phi\phi}$ can again be reduced to quadrature

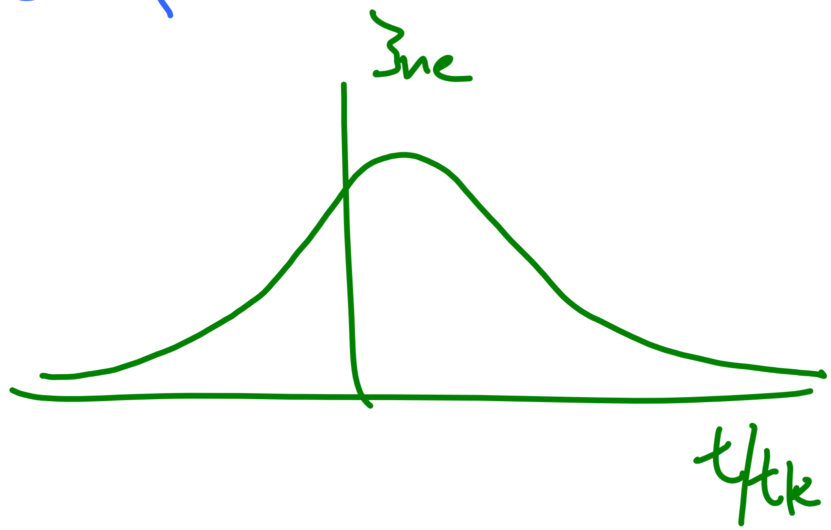


← coarsening

FIG. 4. $\mathcal{G}_{\phi\phi}(\bar{k},\bar{t})$ vs $\bar{k}$ at fixed time slices for the linear TCP. The blue, red and green solid lines are respectively at $\bar{t} = 0.5, 0$ and -0.5. The green dashed line is the correlator if the system were in equilibrium at $\bar{t} = -0.5$. Inset: $\mathcal{L}_{\rm ne}$ vs $\bar{t}$ (solid) and ξ/l_Q (dashed).

Model A CCP $r(t) = -\left(\frac{t}{\tau}\right)^2$

Region $\underline{\text{III}}$ re-equilibrates instead of coarsening

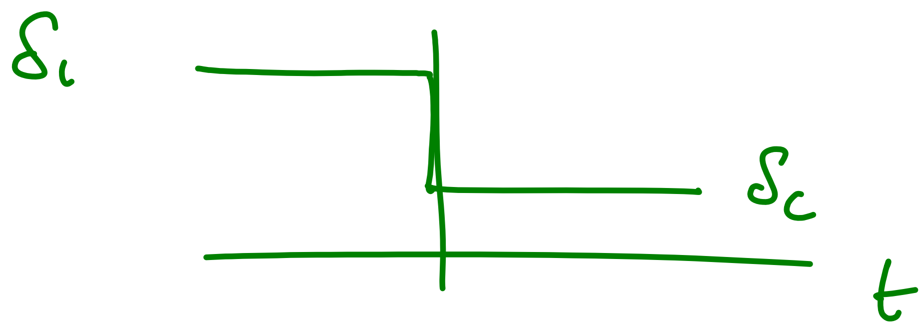


"Thermodynamic" quantities

Model A $\quad f_{ne}(t,\tau) \sim \frac{1}{\ell_{KZ}^{d}} \hat{F}\left(\frac{t}{t_{KZ}}\right)$

Models with energy conservation $\quad \epsilon(t,\tau)$

Can also define a scaling limit for sudden quenches, typically to δ_c



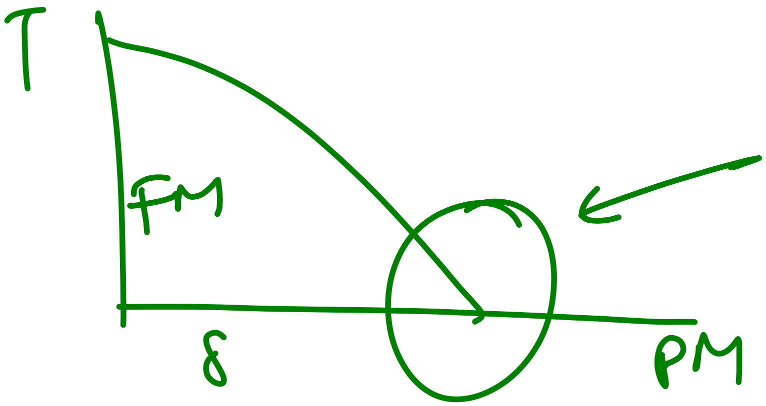
which is an ECP. Focus on equilibration.

Features of the quantum problem

Energy density

Entanglement / Diagonal entropy

$k_B T / (\hbar / t_{KZ})$ is now variable



Both classical and quantum transitions in play

Gaussian scalar QFT $\quad \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - m^2 \phi^2$

Mode expansion $\quad \phi_k(t) = f_k(t)\, a_k + f^*_{-k}(t)\, a^\dagger_{-k}$

Need to solve $\left(\frac{d^2}{dt^2} + \left[k^2 + m^2(t)\right]\right) f_k(t) = 0$

with some conditions.

Construct global solution from matching arbitrary $m^2(t)$ where WKB is valid to exact solution of linear protocol.

$\longrightarrow$ universality, scaling functions

Irrelevance of interactions in $d \geq 4$?

At $t=0$ $\quad \frac{E}{V} = e \sim \frac{1}{\ell_{kz}^{d+z}} \rightsquigarrow \frac{E_{eq}}{V} \sim \left(T^{1/z}\right)^{d+z}$

$\Rightarrow \quad T_{eff} \sim \frac{1}{\ell_{kz}^{z}} \sim \frac{1}{t_k}$.

$\Rightarrow$ scattering time $\propto \frac{1}{T_{eff}} \sim t_k \qquad d < 3+1 \qquad R$

$\propto \frac{\log^2(\Lambda/T_{eff})}{T_{eff}} \sim \log^2(t_k/t_0) \cdot t_k \qquad d = 3+1 \qquad I$

$\propto \frac{1}{T_{eff}^{1+2\epsilon}} \sim t_k^{1+2\epsilon} \qquad d = 3+1+\epsilon \qquad I$

Damle, Sachdev

QBE

Scattering is dangerously irrelevant

$$\frac{t_{scatt}}{t_{KZ}} \to \infty \quad \text{in scaling limit}$$

<u>but</u> cannot be dropped at all times.

Defect densities?

For defects of dimension p with characteristic separation ξ_{ne} along any hyperplane of co-dimension $d-p$

$$\rho_{defect} \sim \frac{1}{\xi_{ne}^{d-p}} \sim \frac{1}{\ell_{KZ}^{d-p}} f\left(\frac{t}{t_{KZ}}\right)$$

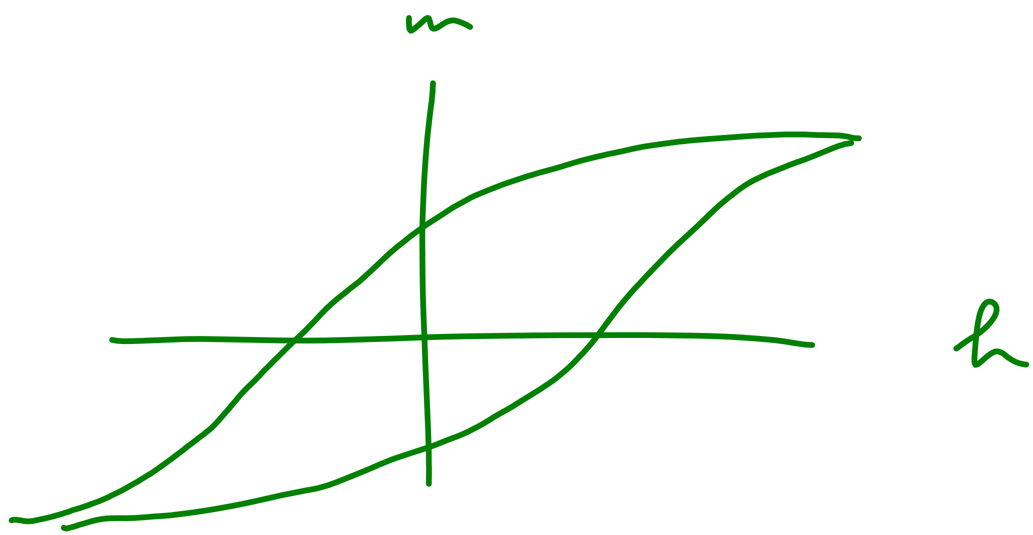
If we hold t/t_{KZ} fixed

$$\rho_{defect} \sim \frac{1}{\ell_{KZ}^{d-p}} \sim \left(\frac{1}{\tau}\right)^{\frac{a\nu(d-p)}{a\nu z+1}}$$

$\leftarrow$ velocity

Varying h at $\delta = \delta_c$

Universal hysteresis curve



Would be good to measure.

Magnets?
Liquid gas?

KZ without symmetry breaking

Consider continuous transitions between a topologically ordered phase and a phase with relatively less or no topological order.

Topologically ordered phase : Emergent gauge field
No local OP

Consider Z_2 topological order - physics of deconfined phase in Z_2 gauge theory

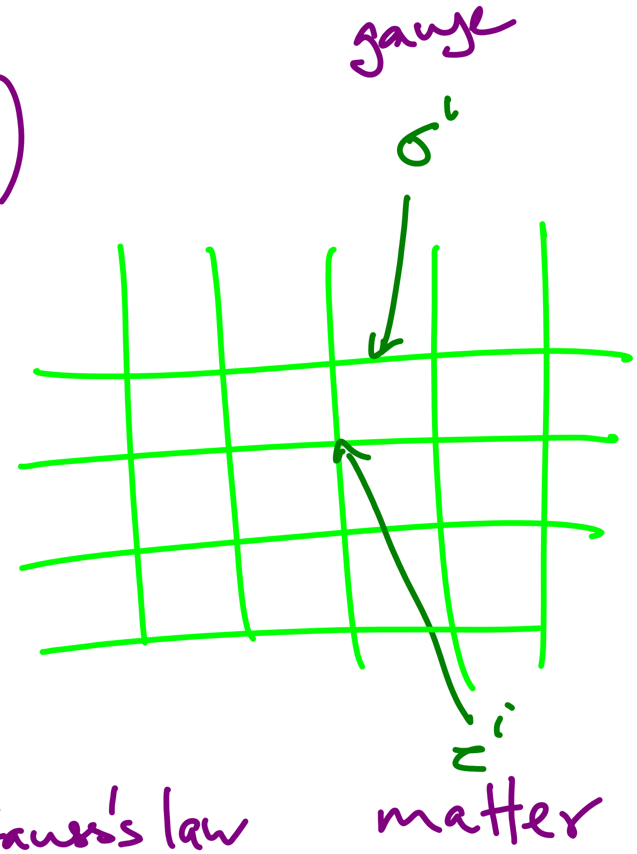
Simplest to consider the $\mathbb{Z}_2$ GT with matter

$\equiv$ perturbed toric code

Toric Code (Kitaev 1997)

$$H = -K \sum_p \left(\prod_p \sigma^z \right) \cancel{- \Gamma \sum_s \sigma^x}$$

$$\cancel{-J \sum_b \tau^z_s \sigma^z_b \tau^z_{s'}} \; - \Gamma_M \sum_s \tau^x_s$$



Constraint $\quad \tau^x_s \prod_{+} \sigma^x_b = \mathbb{1}$ Gauss's law

Kitaev's insight – get rid of matter

$$H = -K \sum_p \prod_p \sigma^z - h \sum_s \prod_+ \sigma_b^x$$

to get spin only model with Z_2 topological order

plaquette excitations – visons

site excitations – spinons

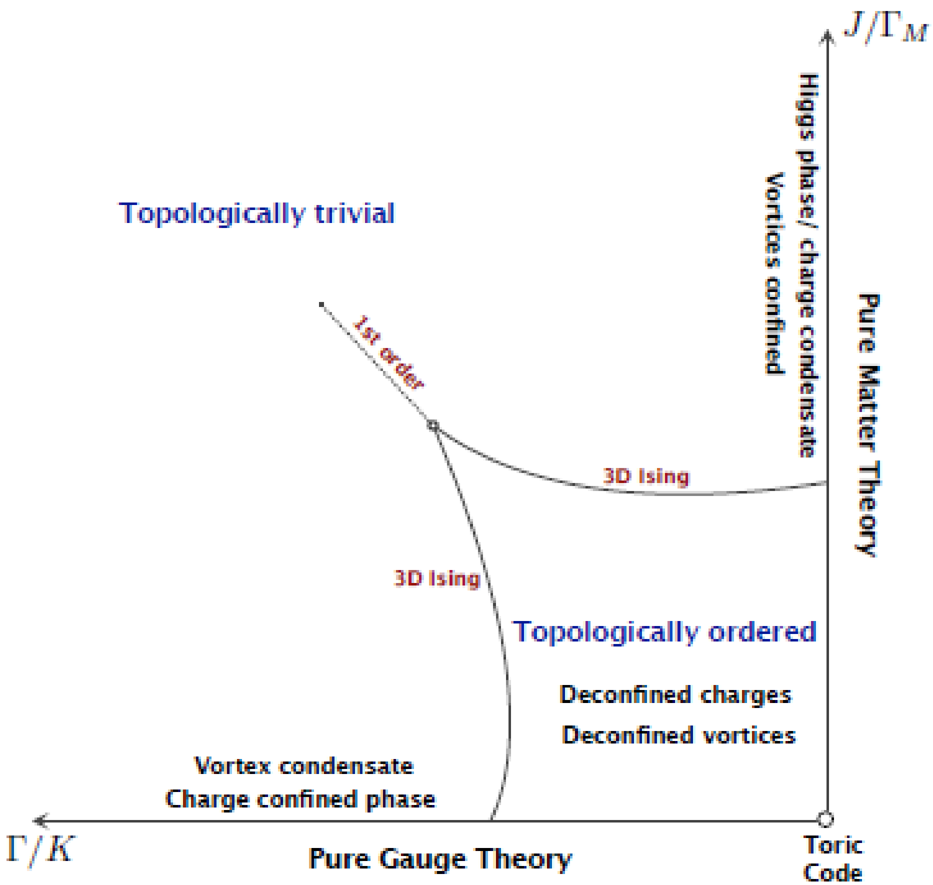


FIG. 2. $T = 0$ phase diagram of the $\mathbb{Z}_2$ theory in $d = 2$ dimensions. The matter and gauge axes are dual, and the Higgs and charge confined phases are smoothly connected.

Consider general TCP between topological and non-topological phases

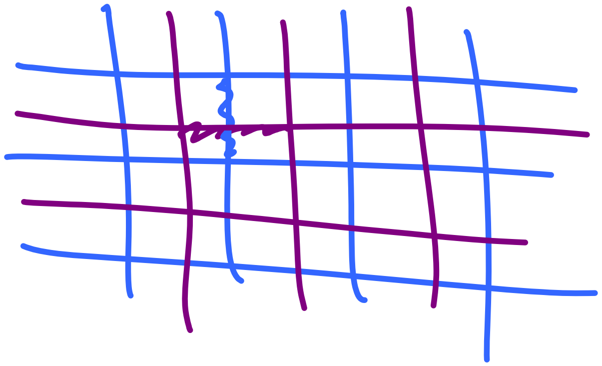
1. Along the matter axis – TCP in the TFIM (classic KZ)

In 2+1 TFIM has ordered phase at $T>0$ so TCP leads to coarsening at late times

(Subtleties – see paper)

2. Along the pure gauge axis — can understand the behavior of gauge invariant observables by duality.

Duality



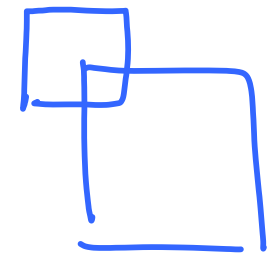
$\sim$ $\tau^z_i \tau^z_j = -1$

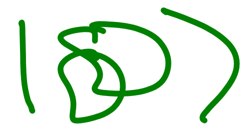
$\sim$ $\sigma^x_{ij} = -1$

$\Rightarrow \prod_+ \sigma^x_{ij} = 1$

τ^x flips spin $\quad \prod \sigma^z_{ij}$ flips surrounding plaquette

Constraint $\Rightarrow$ closed loops (domain walls)



FM	PM	
Confined	Deconfined	Γ/J
$\lvert \circ \ \circ \rangle$	$\sum$ 	

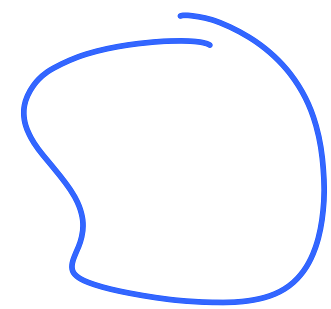
Changing $\frac{\Gamma}{J}$ introduces string tension for the condensed loops

TCP causes condensed loops to thin out – ending in a long time coarsening process we term "string-net coarsening"

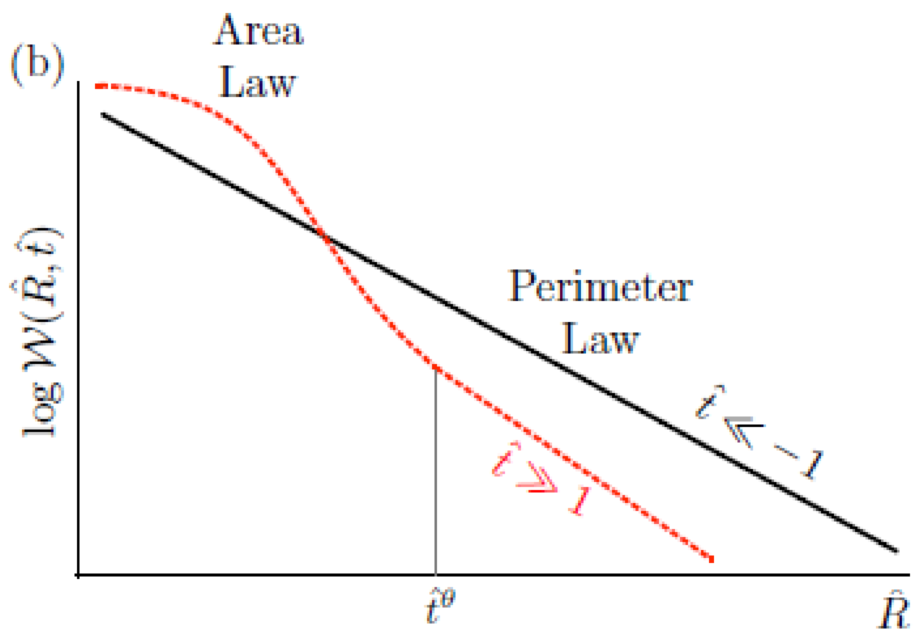
(trade E^2 for B^2)

Scaling forms for thermodynamic quantities

Wilson loop $\quad W[c] = \prod_{b \in c} \sigma^x_b$



c



Horizon effect: State looks topologically ordered on longest length scales

(Also: amplification of topological degeneracy)

3. Generic path (no longer dual to broken symmetry problem)

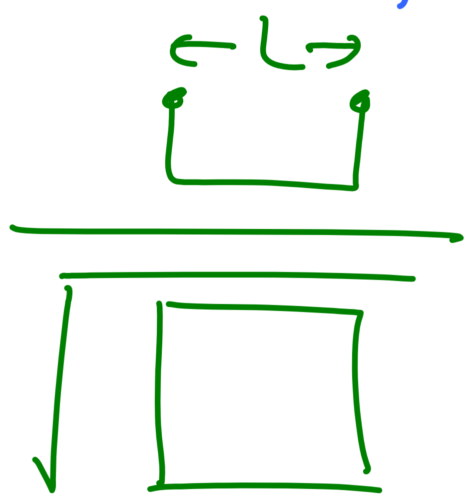
Off the gauge axis matter is gapped but excited at late times upon thermalization:

$$\ell_{\text{matter}} \sim e^{+\Delta/T_{\text{eff}}} \gg \ell_{KZ}$$

$\Rightarrow$ matter coupling is dangerously irrelevant since it terminates coarsening

Study thermodynamic observables

Fredenhagen-Marcu order parameter

$FM \sim$ 

$\propto \begin{cases} O(L^0) & \text{non top} \\ O(e^{-L}) & \text{topological} \end{cases}$

Similar story elsewhere in plane.

All topological phases on lattice have some "string-net" structure (flux networks).

Have shown SNC for some transitions out of $SU(2)_k$ phases.

Idea should be applicable quite generally.

cf Dirac strings in spin ice

In closing

New set of theoretical challenges near critical points

- ϵ- expansion, $1/N$ corrections, RG
- AdS / CFT (unpublished)
- coarsening in quantum systems
- Experiments